Theorems in Color

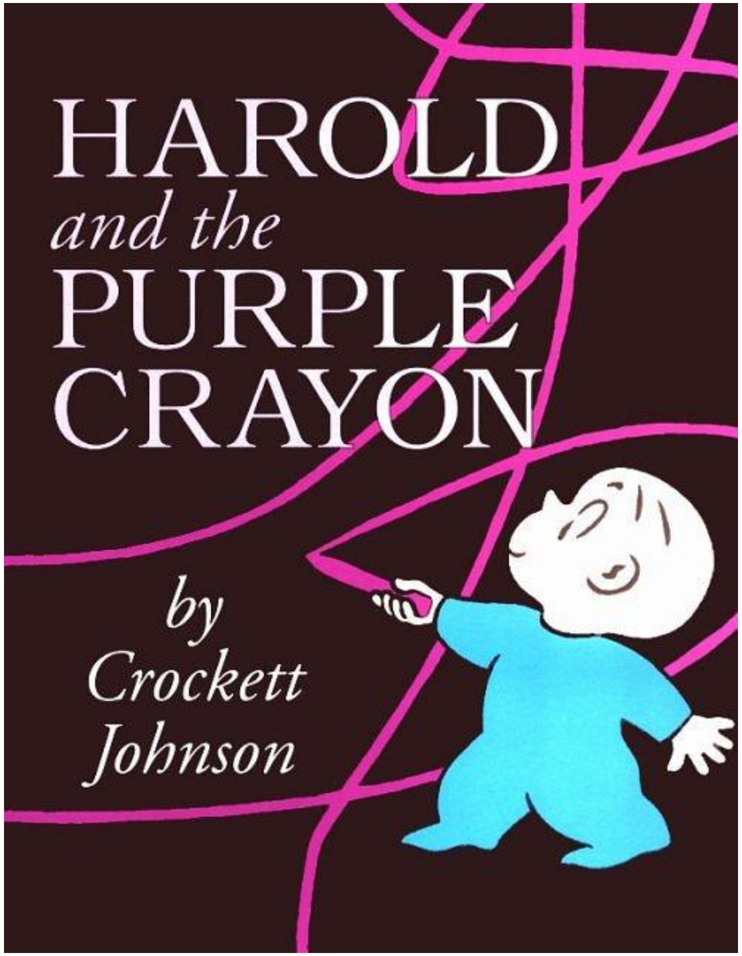
Katie Ambruso Acker, Ph.D.
Robert McGee

Cabrini College

Biography

- **1906 born David Johnson Leisk**
- **Education**
  - 1924 Cooper Union
  - 1925 New York University
- **Cartoons**
  - 1934-1940 New Masses
  - 1940-1943 Colliers, The Little Man with Eyes
  - 1942-1953 Barnaby
- **1940 married Ruth Krauss, author**
- **Children's Books**
  - 1952-1965 Harold and the Purple Crayon
- **Mathematics Paintings**
- **1975 Dies of lung cancer**

Harold and the Purple Crayon



Crockett Johnson on His Paintings

"In my geometric paintings, I use, as intrinsic tools the mathematical geometry and the mathematical methods I, as a desultory and very late scholar, have been able to absorb."

C. Johnson, On the Mathematics of Geometry in my Abstract Paintings, *Leonardo* ***5****, 1972.*

Acknowledgments

- The Crockett Johnson Homepage
  - http://www.ksu.edu/english/nelp/purple/

- Peggy Kidwell
  - Curator of Mathematics, Smithsonian Institute.
  - We wish to thank the Smithsonian Institute for allowing us to use digital images of Crockett Johnson's paintings.

Pythagorean Theorem

Proposition 47 Euclid Book I

- In a right triangle, the square on the hypotenuse is equal in area to the sum of the squares on the sides.

- There are over 300 proofs of the Pythagorean theorem.

$$a^2 + b^2 = c^2$$

Pythagorean Theorem

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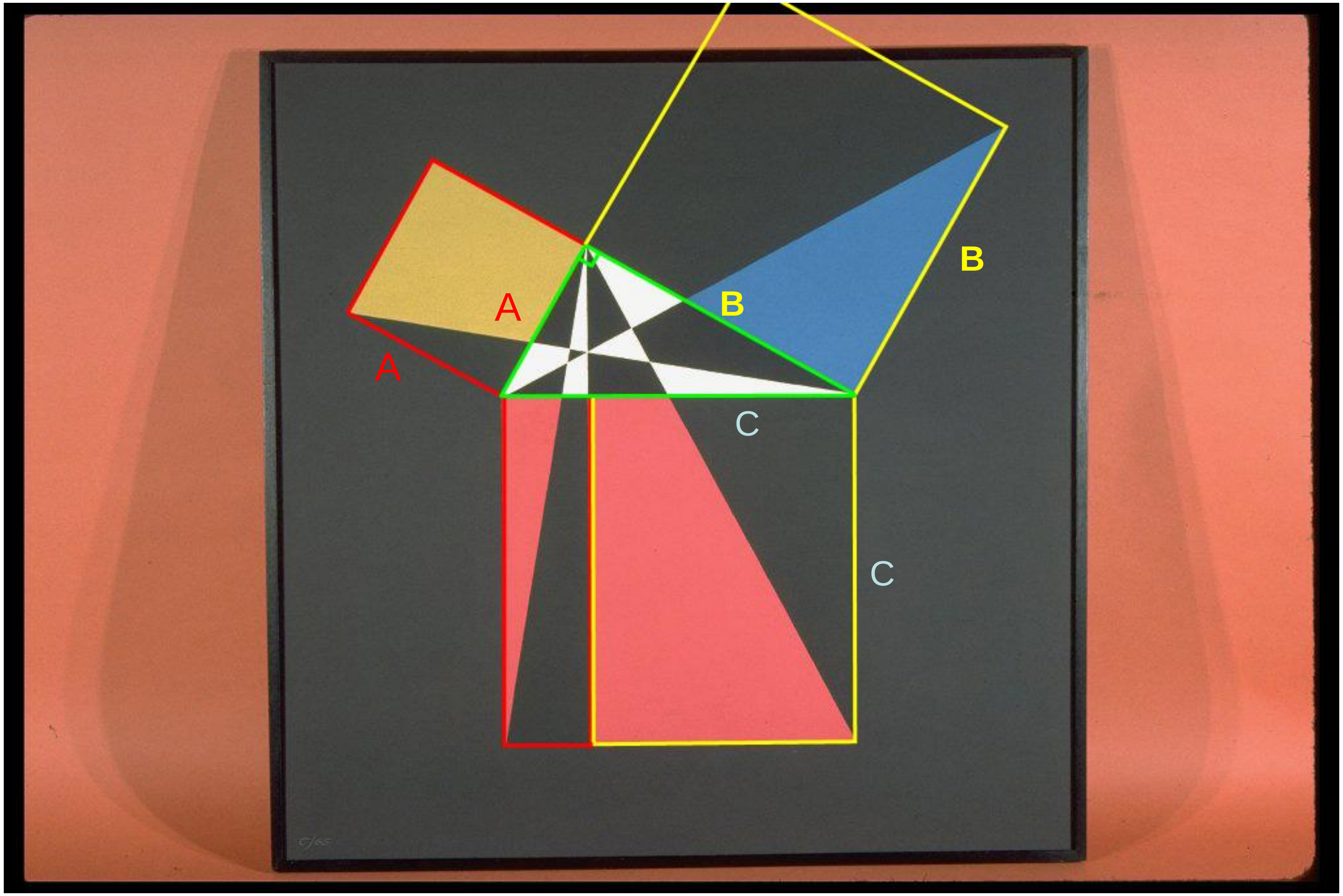
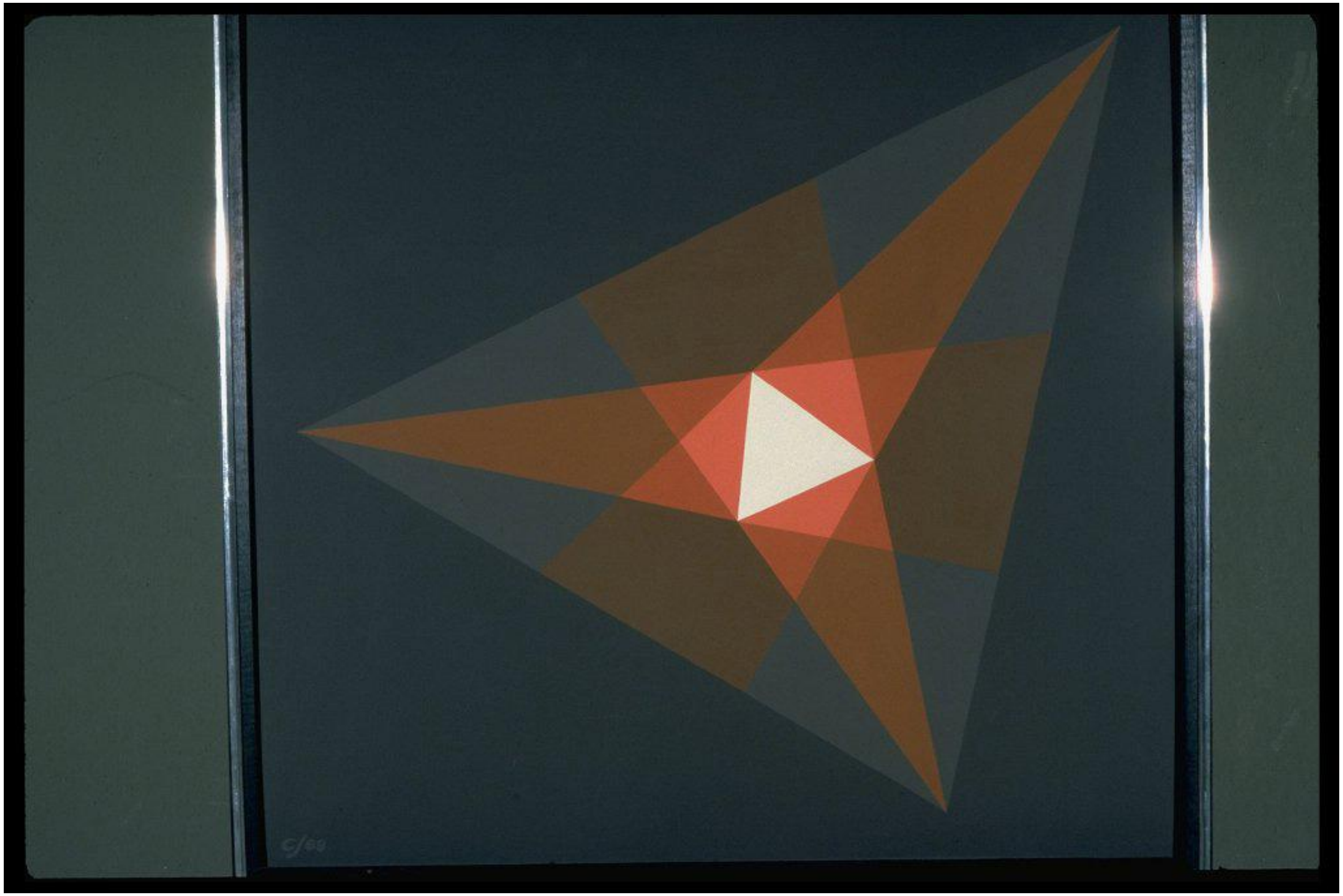


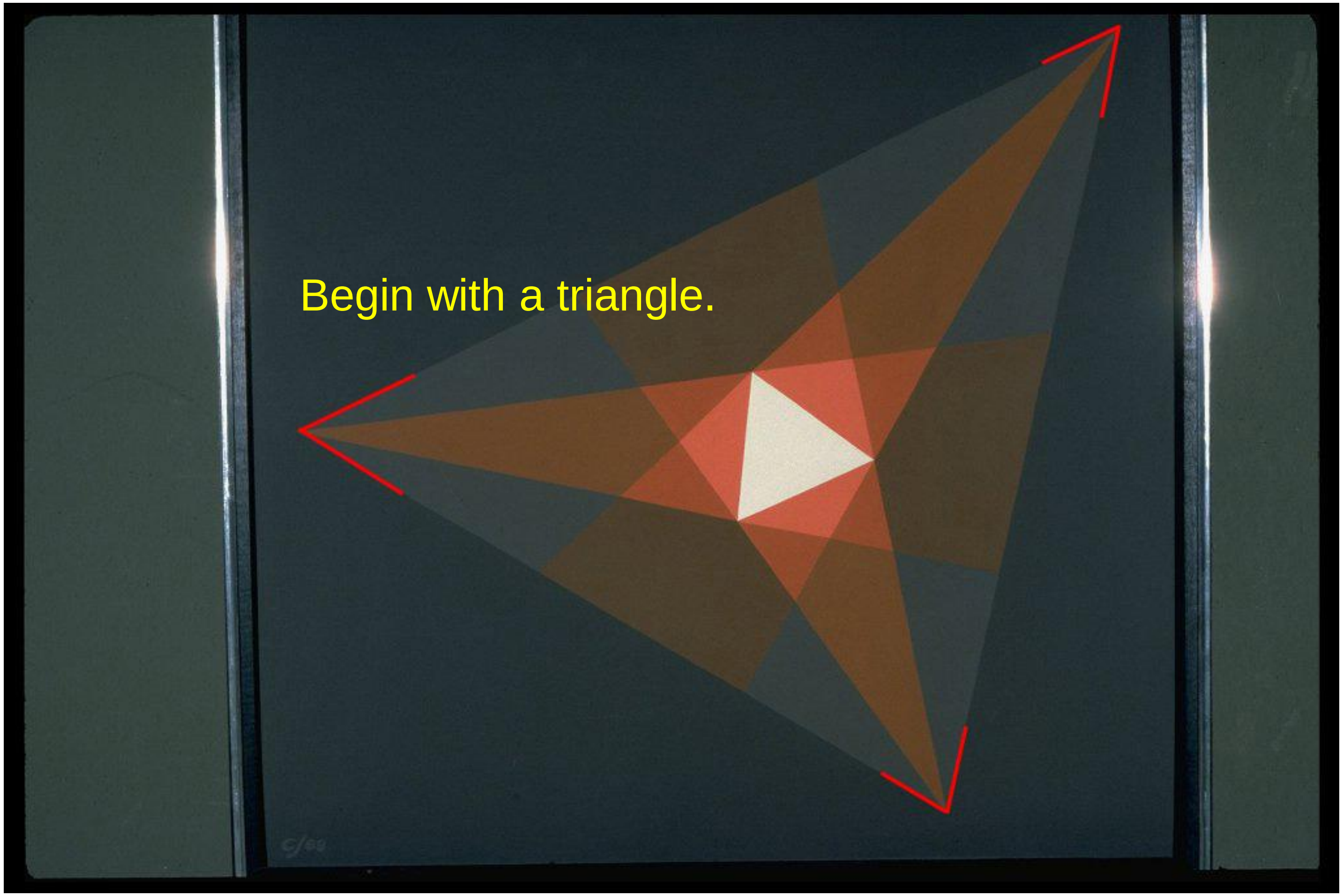
Morley Triangle

Corresponding angle trisectors meet at the vertices of an equilateral triangle.

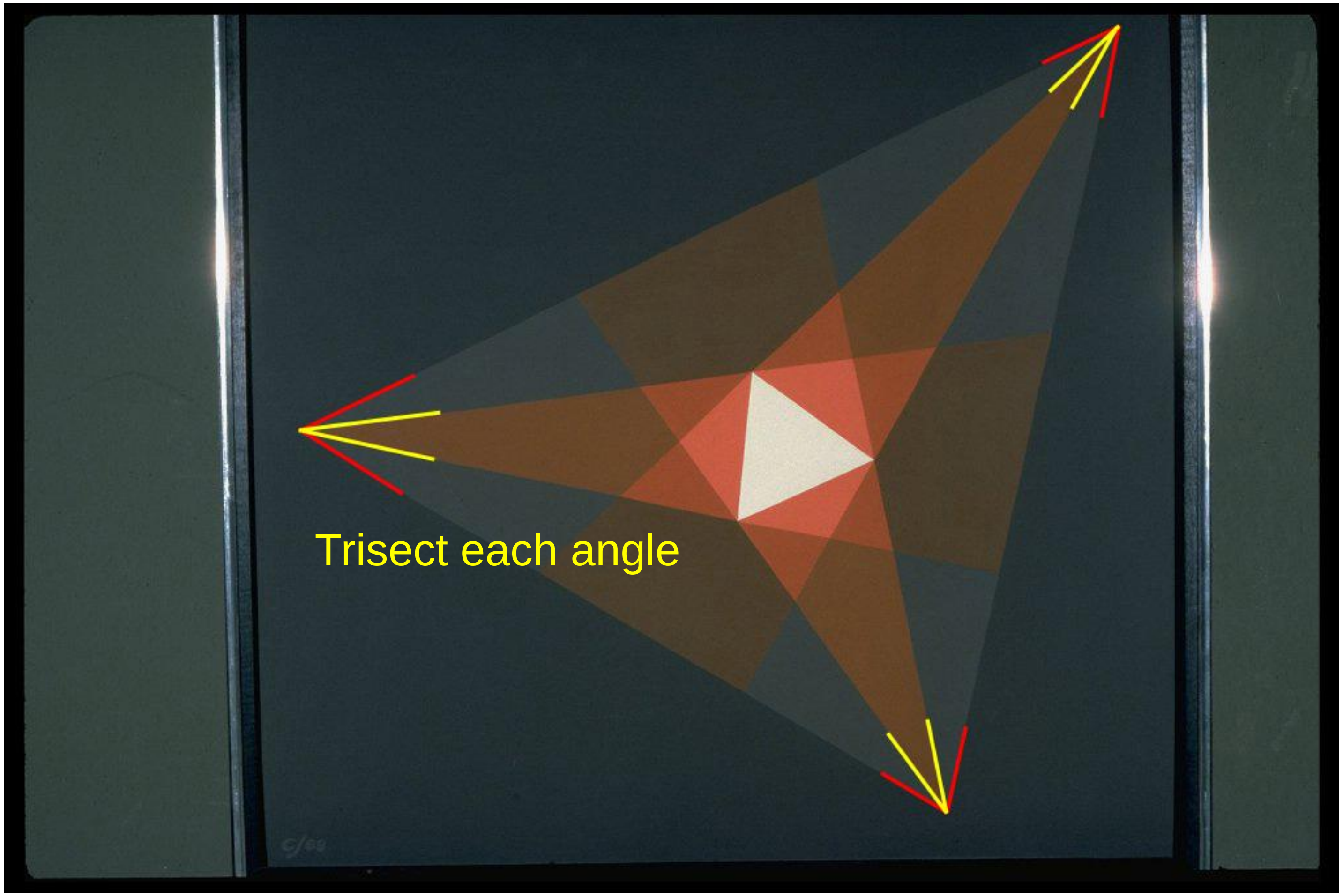
Morley Triangle



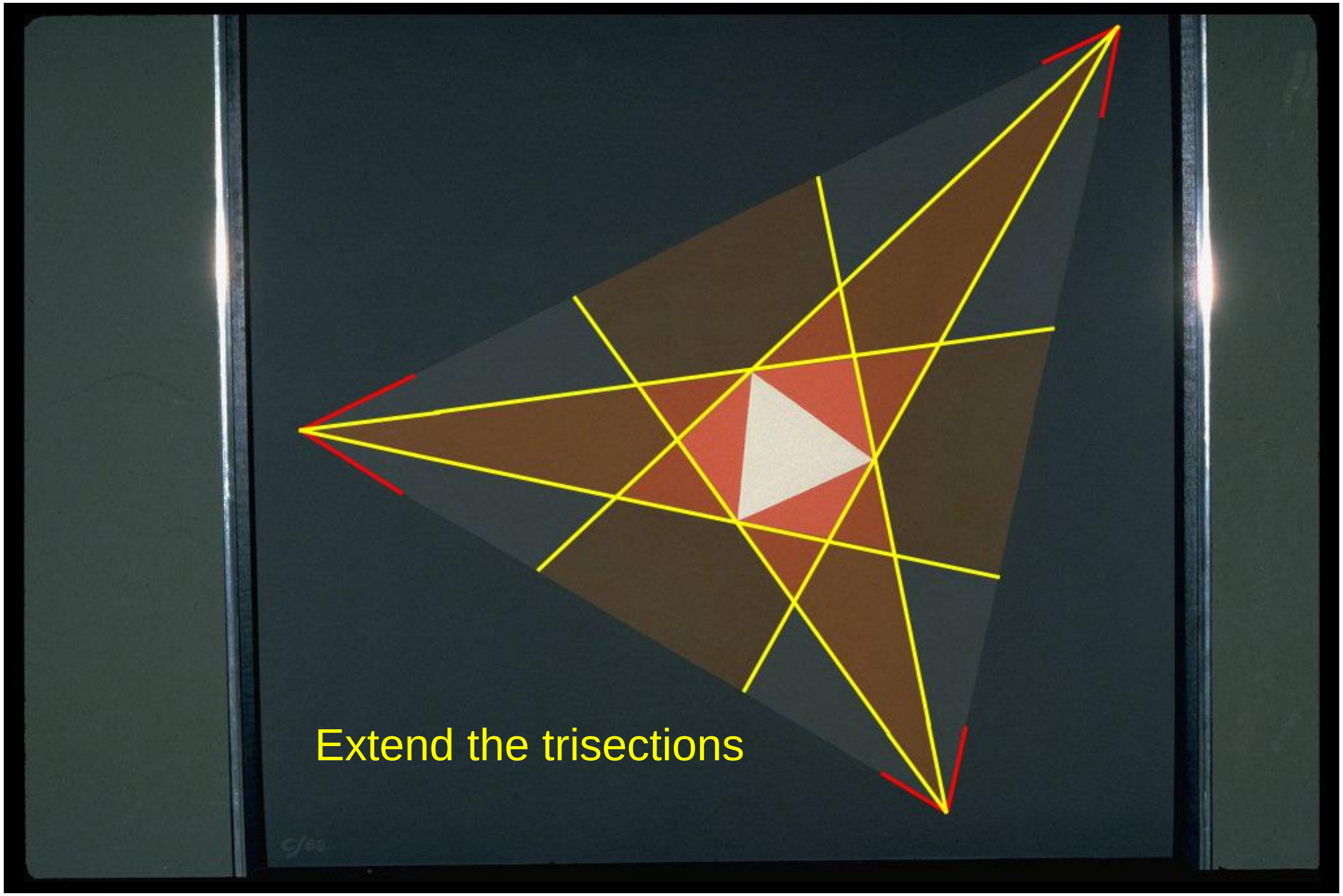
Morley Triangle



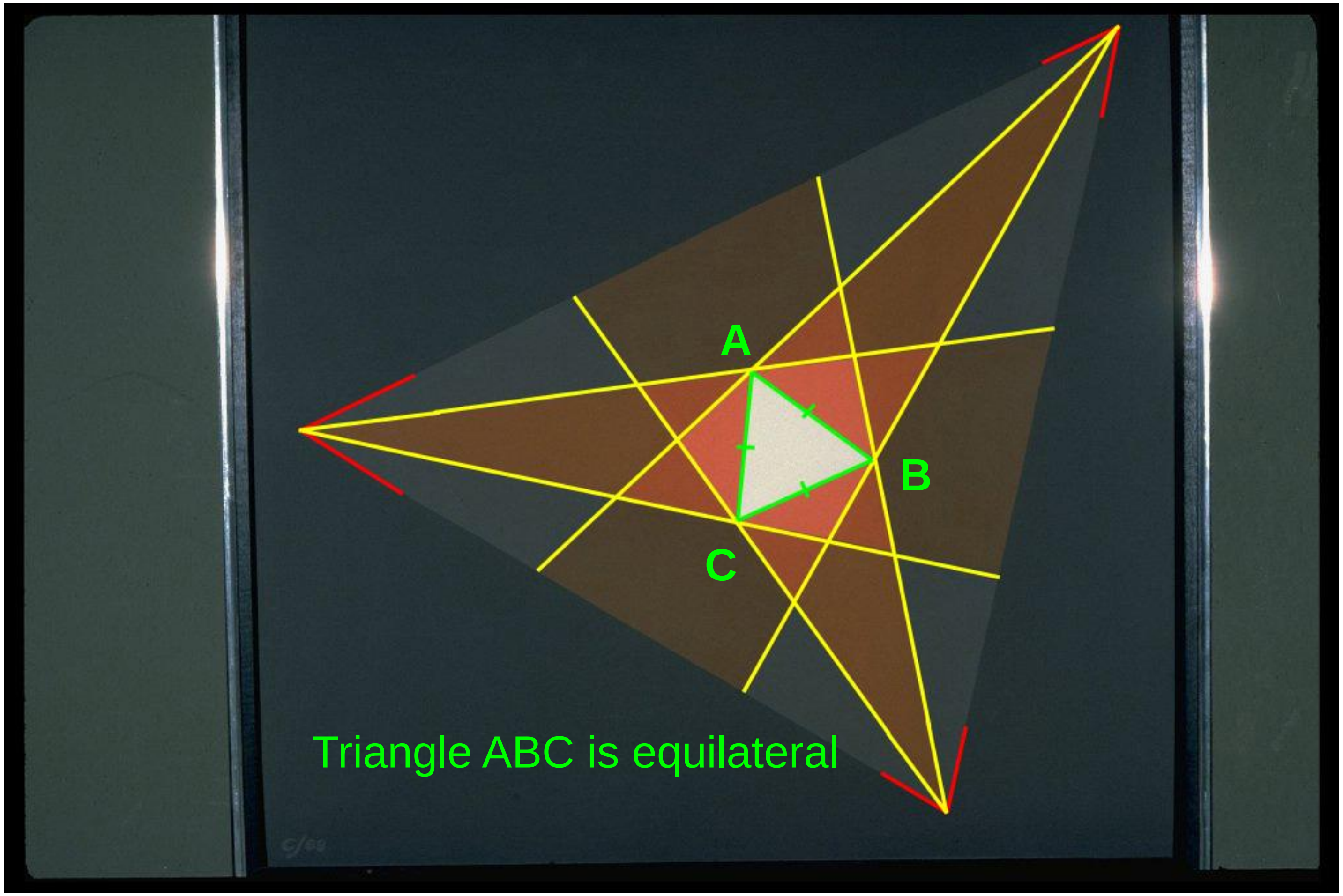
Morley Triangle



Morley Triangle



Morley Triangle



Nine-Point Circle

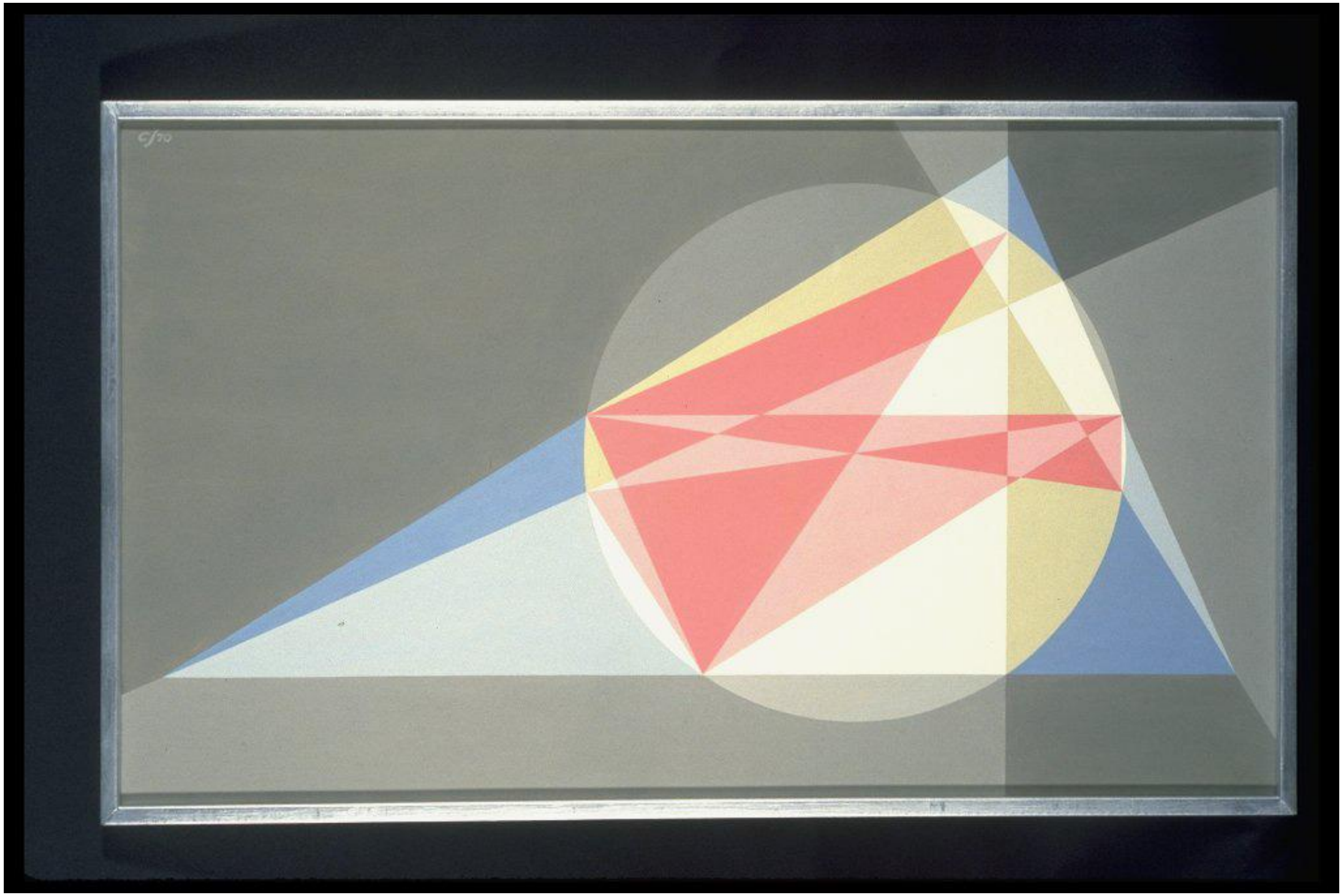
3 points located at the midpoints of the sides of a triangle.

3 points from the feet of the altitudes from each side.

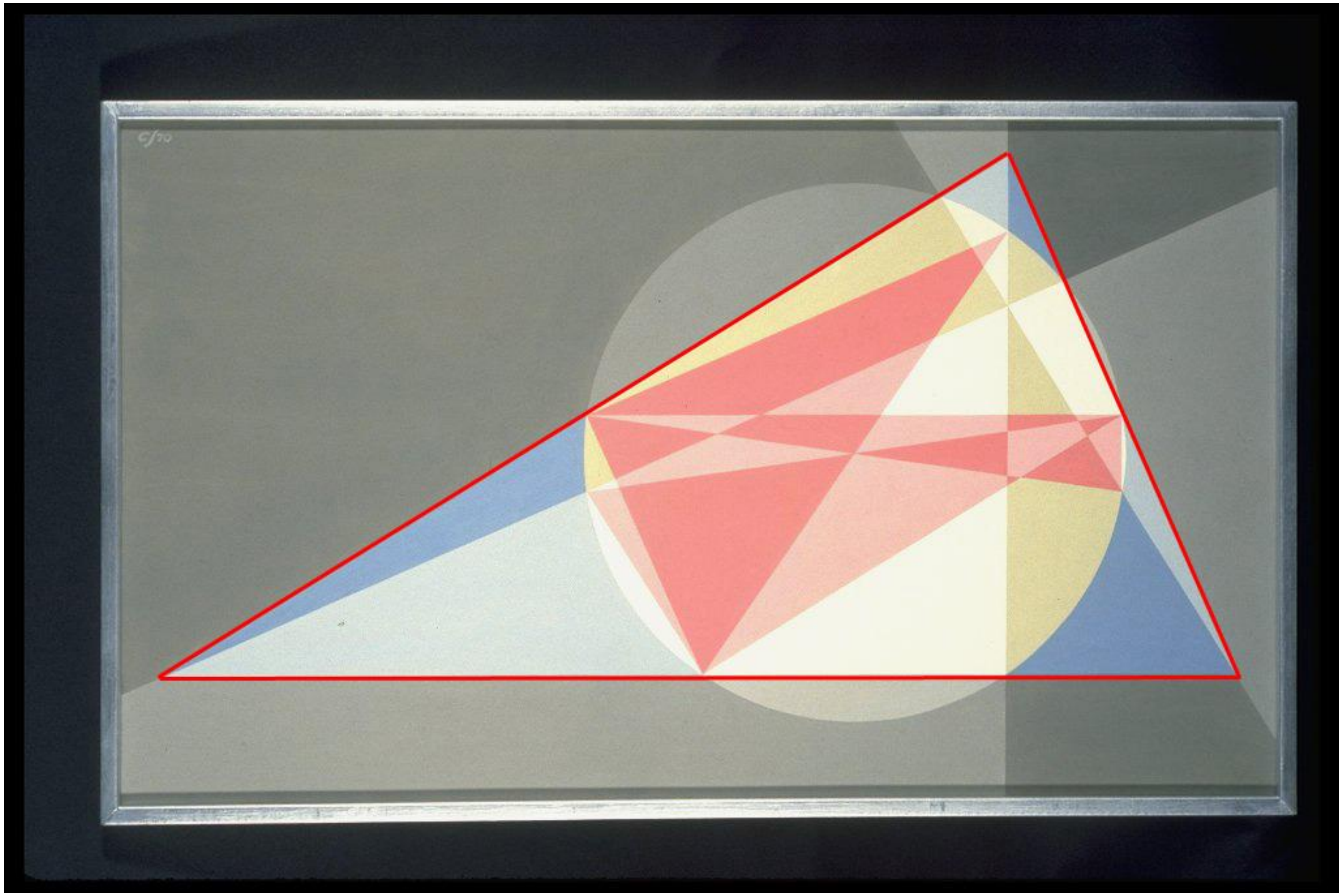
3 points located at the midpoints between the orthocenter and the vertices of the triangle.

Orthocenter-the point of concurrency of the altitudes of a triangle.

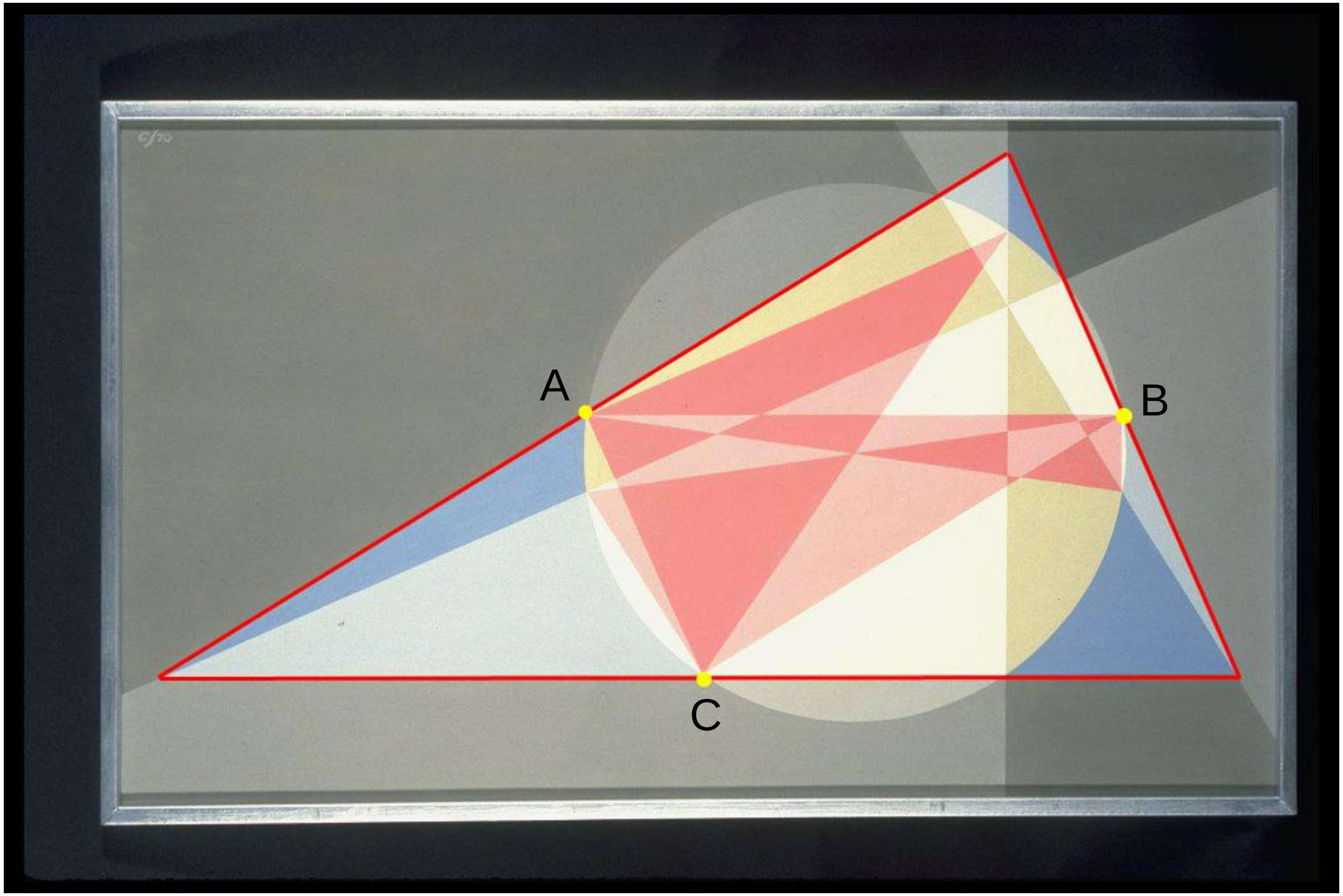
Nine-Point Circle



Nine-Point Circle

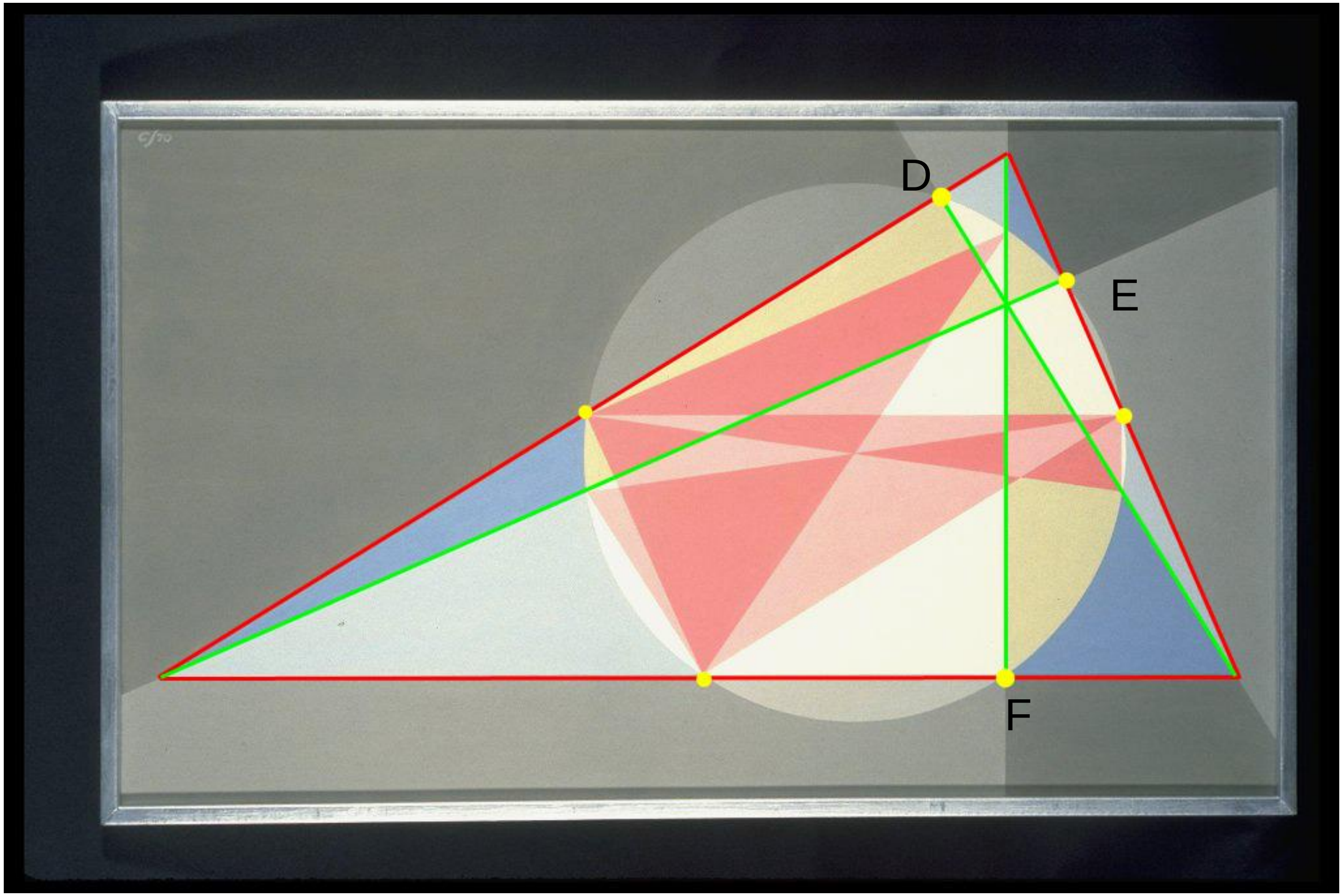


Nine-Point Circle



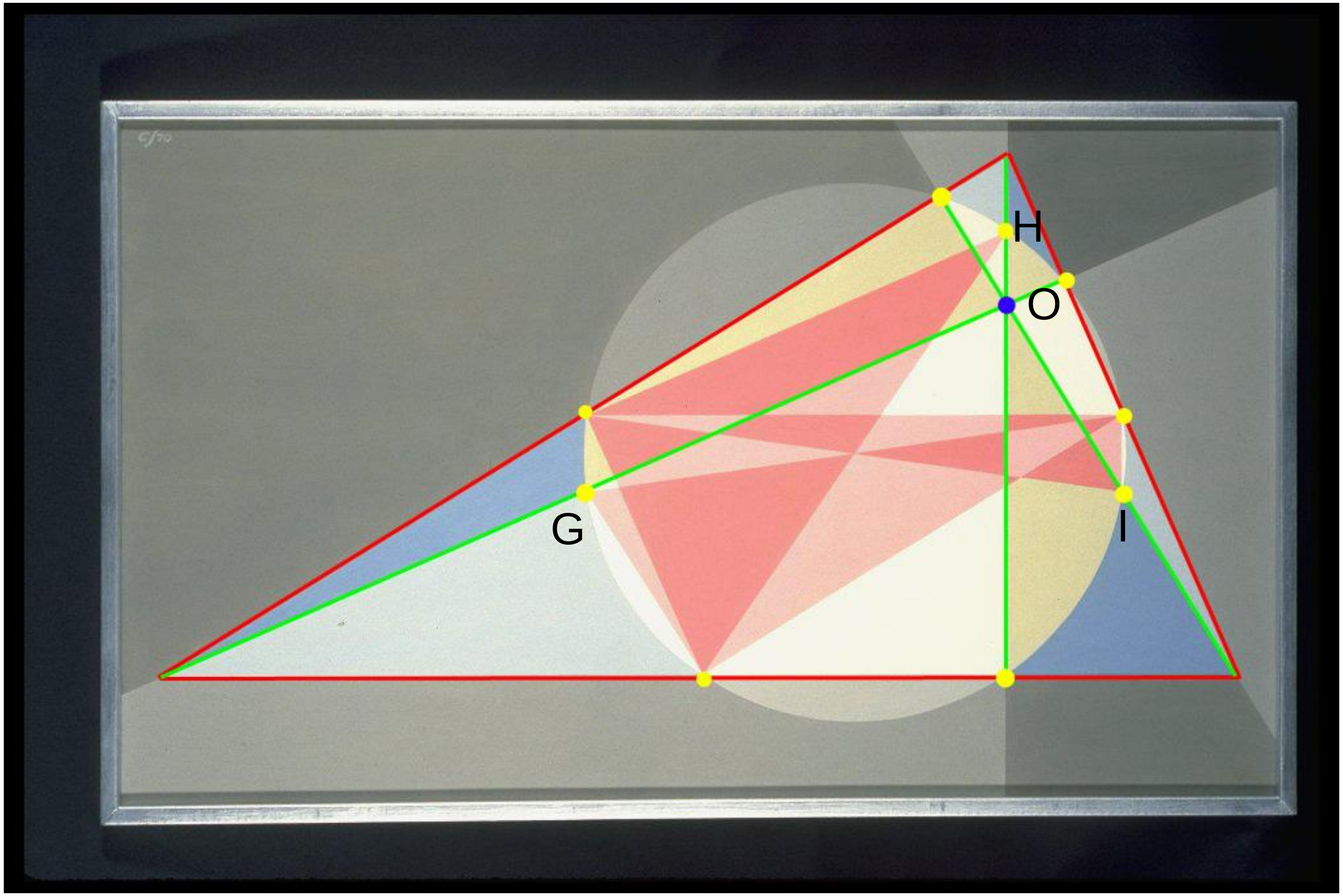
A, B, and C are the midpoints of the sides of the triangle.

Nine-Point Circle



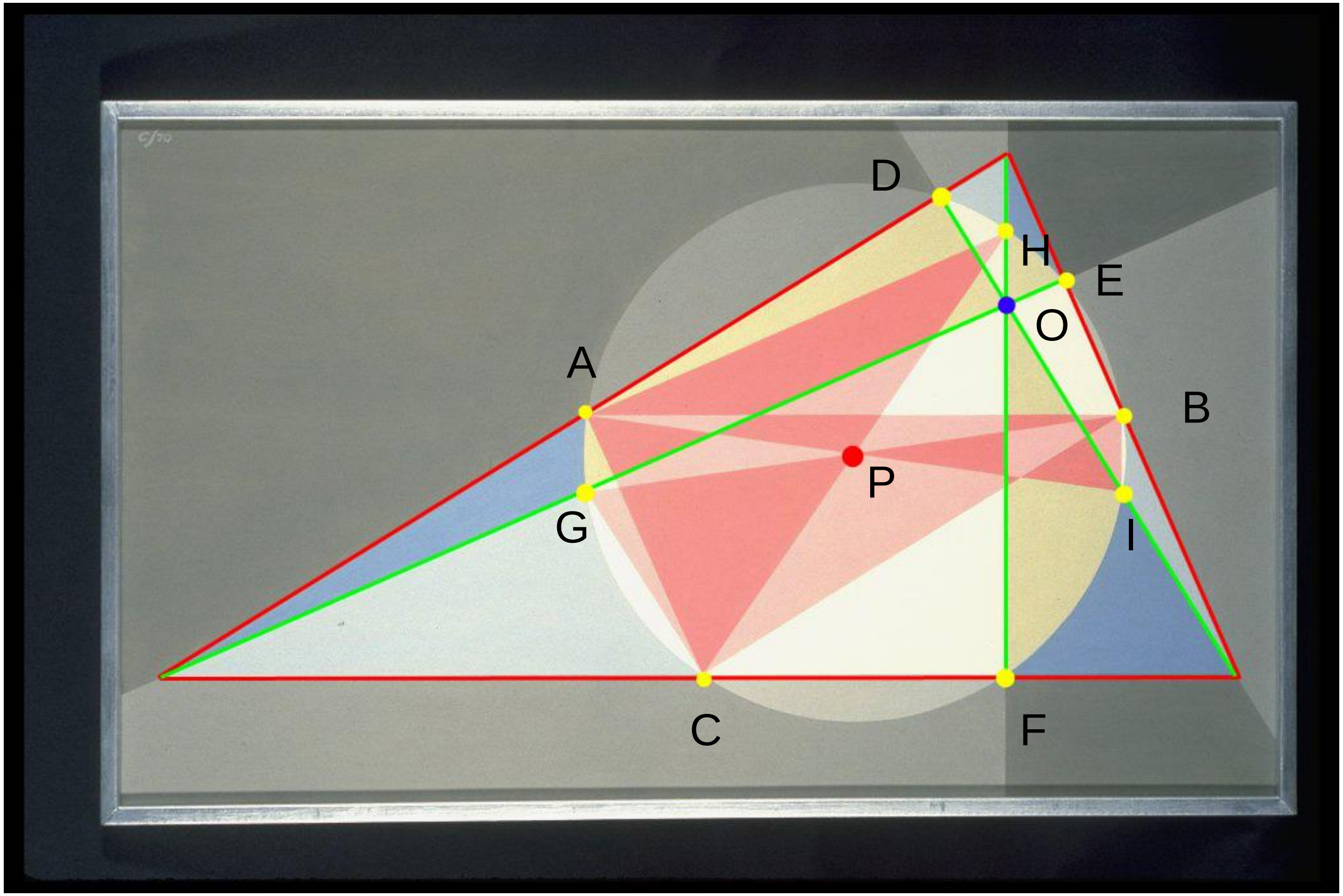
D,E, F are the feet of the altitudes.

Nine-Point Circle

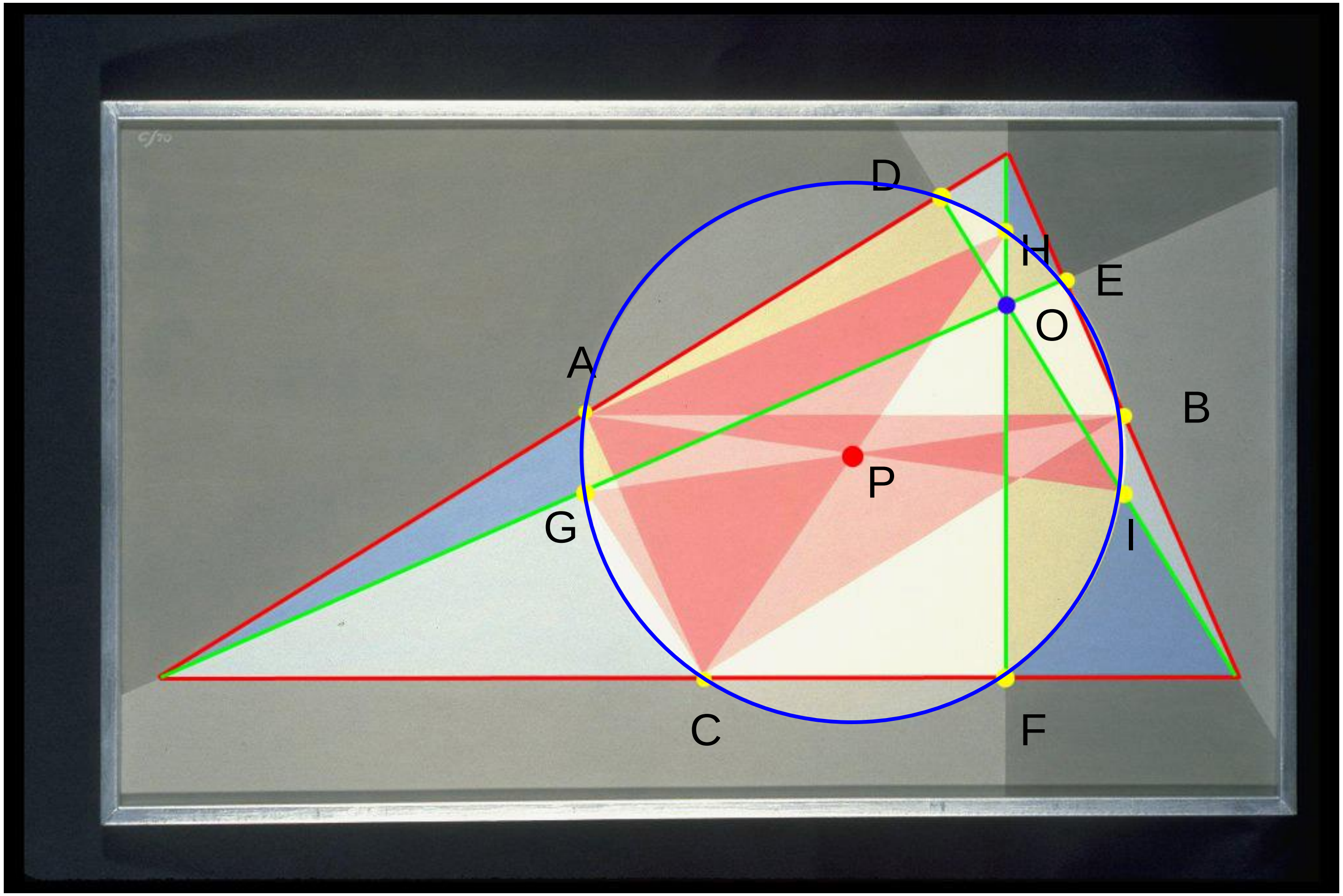


G, H, I are the midpoints between O, the orthocenter, and the vertices of the triangle. These points are also known as Euler Points.

Nine-Point Circle



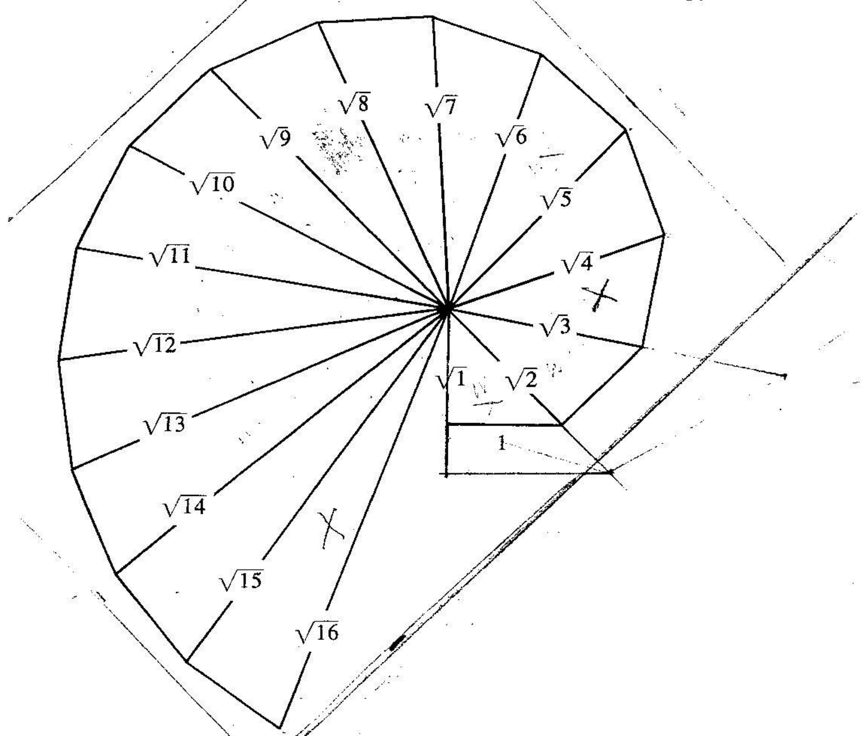
Nine-Point Circle



Theodorus

- Thought to be a Pythagorean.
- Mathematics tutor to Theaetetus and Plato.
- According to Plato, Theodorus was the first to show that square roots of nonsquare integers from 3 to 17 are incommensurable with 1.

E. Valens, p. 61

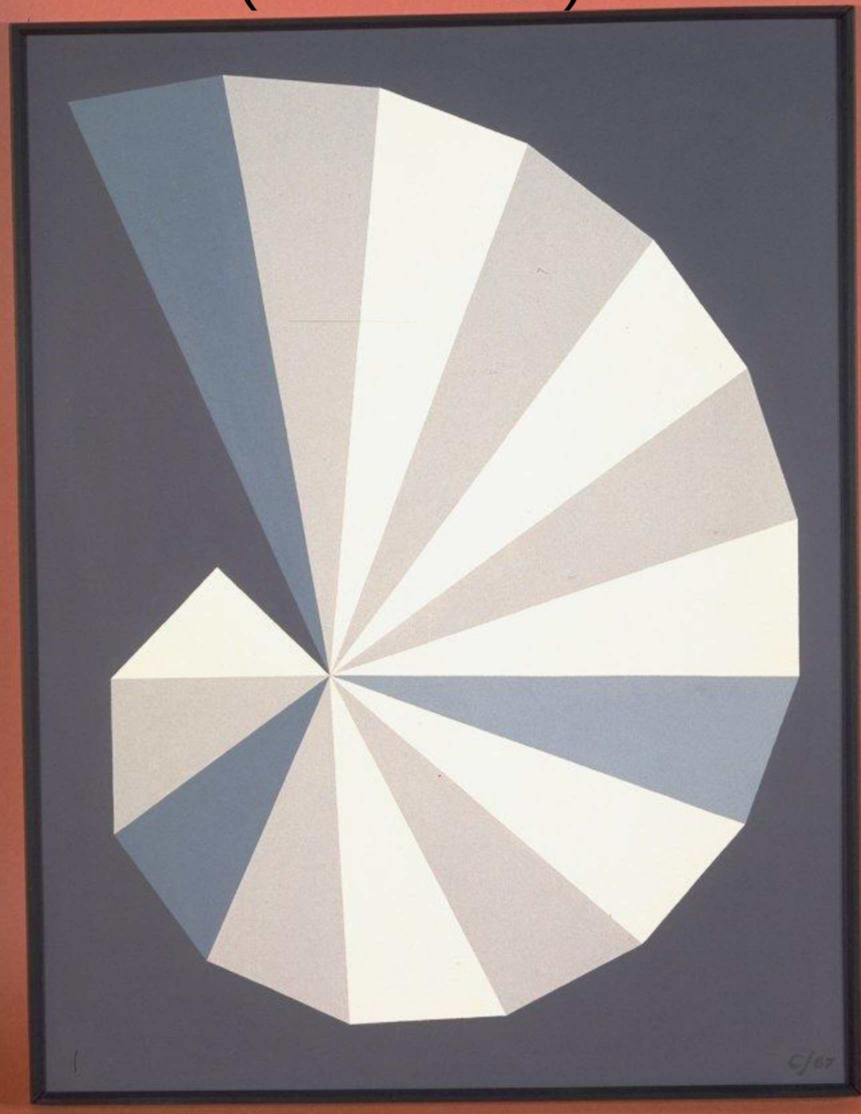


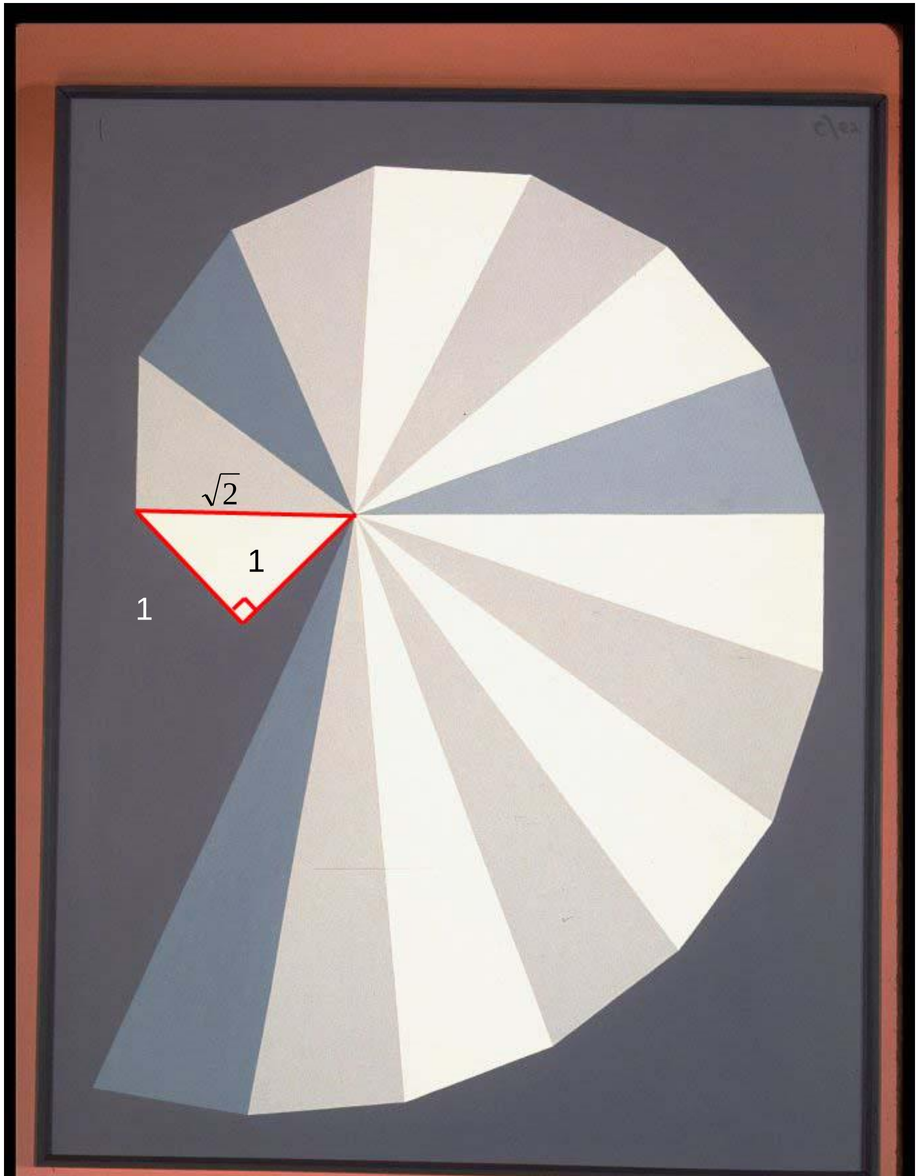
Each new side is derived from the previous side by the Pythagorean theorem. For example, $(\sqrt{14})^2 + 1^2 = (\sqrt{15})^2$.

The first three sides—$\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$—are of particular interest because they are, respectively, the side, the surface diagonal, and the interior diagonal of a unit cube.

Why did Theodorus stop short of the square root of 17? Plutarch said the Pythagoreans "have a horror for the number 17" because it lies between two somewhat magical numbers: 16, which is a square with an area equal to its perimeter, and 18, which is the double of a square and is also a rectangle (3 × 6) with an area equal to its perimeter.

Square roots to 16
(Theodorus)





Square roots to 16
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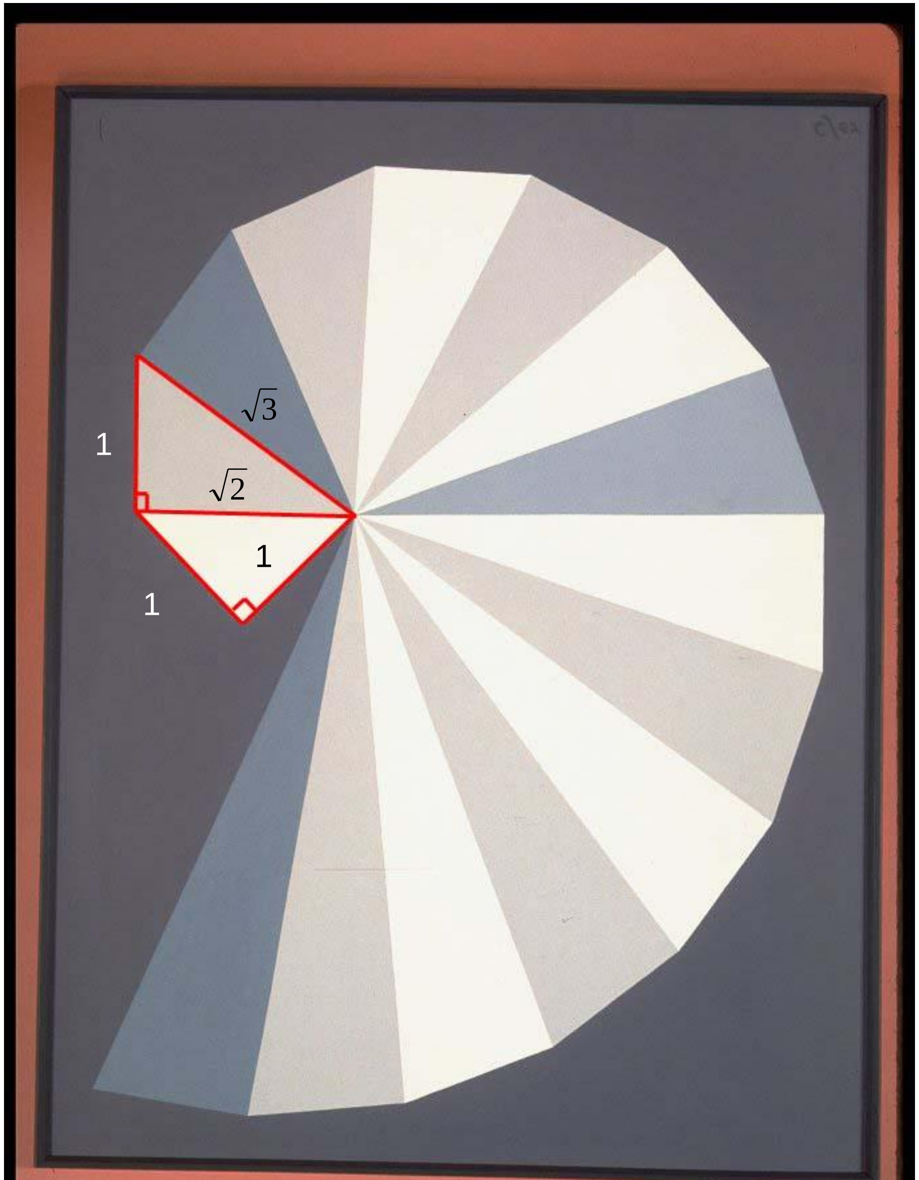

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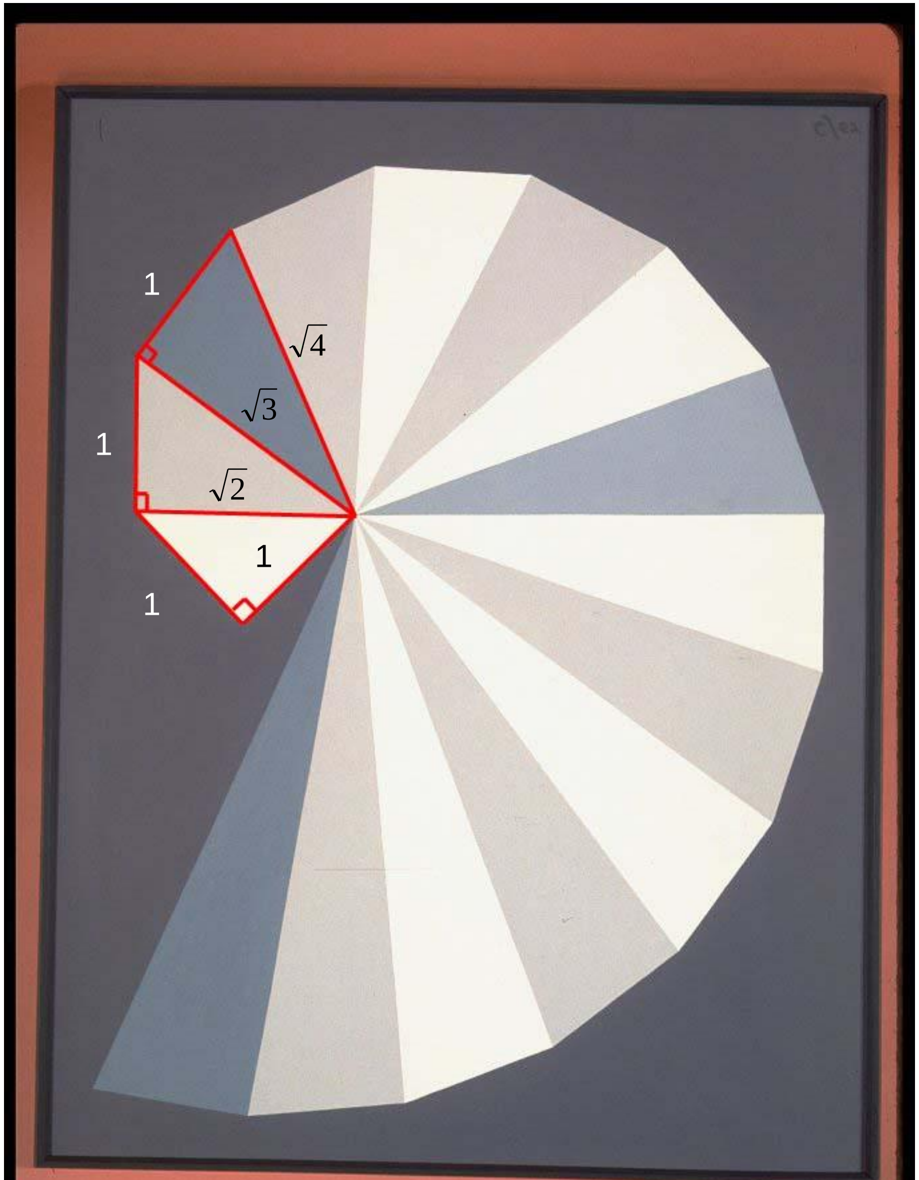

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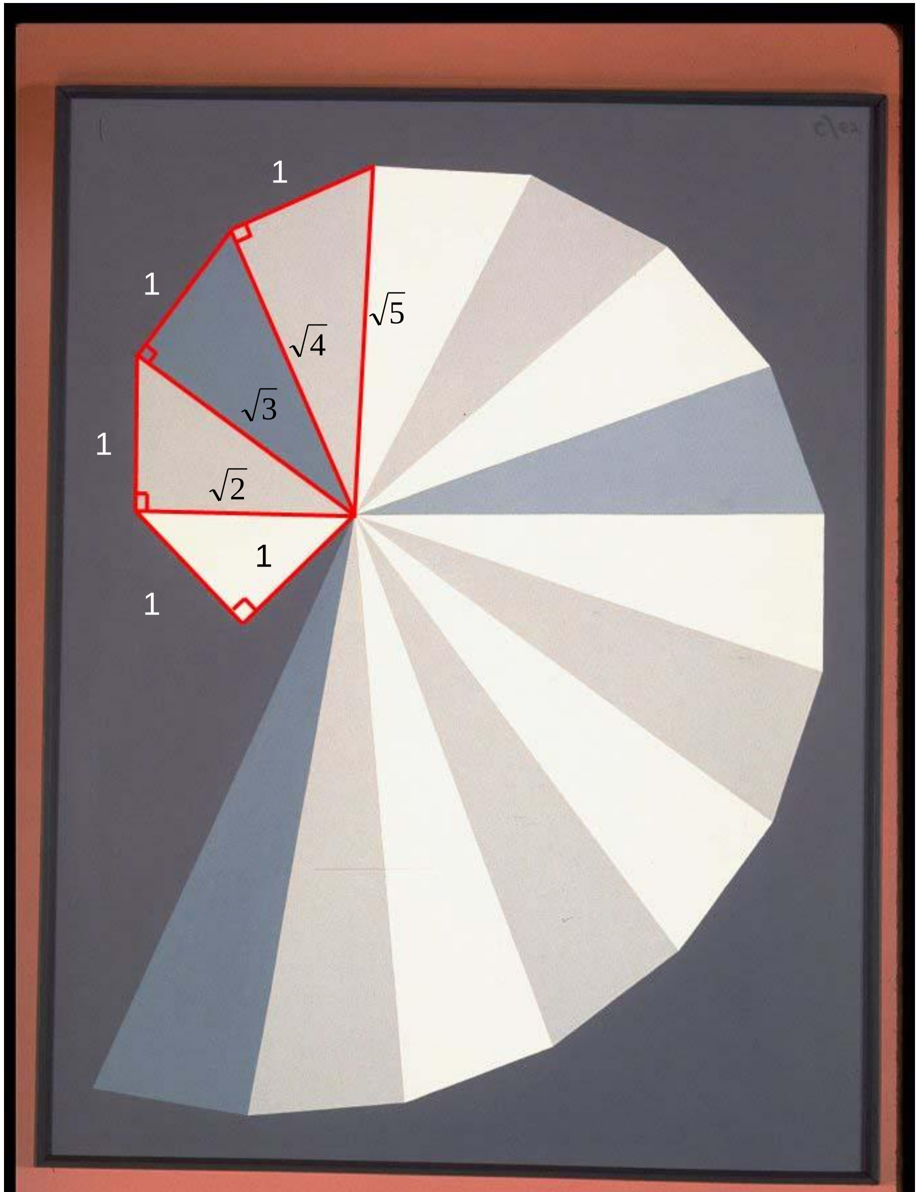

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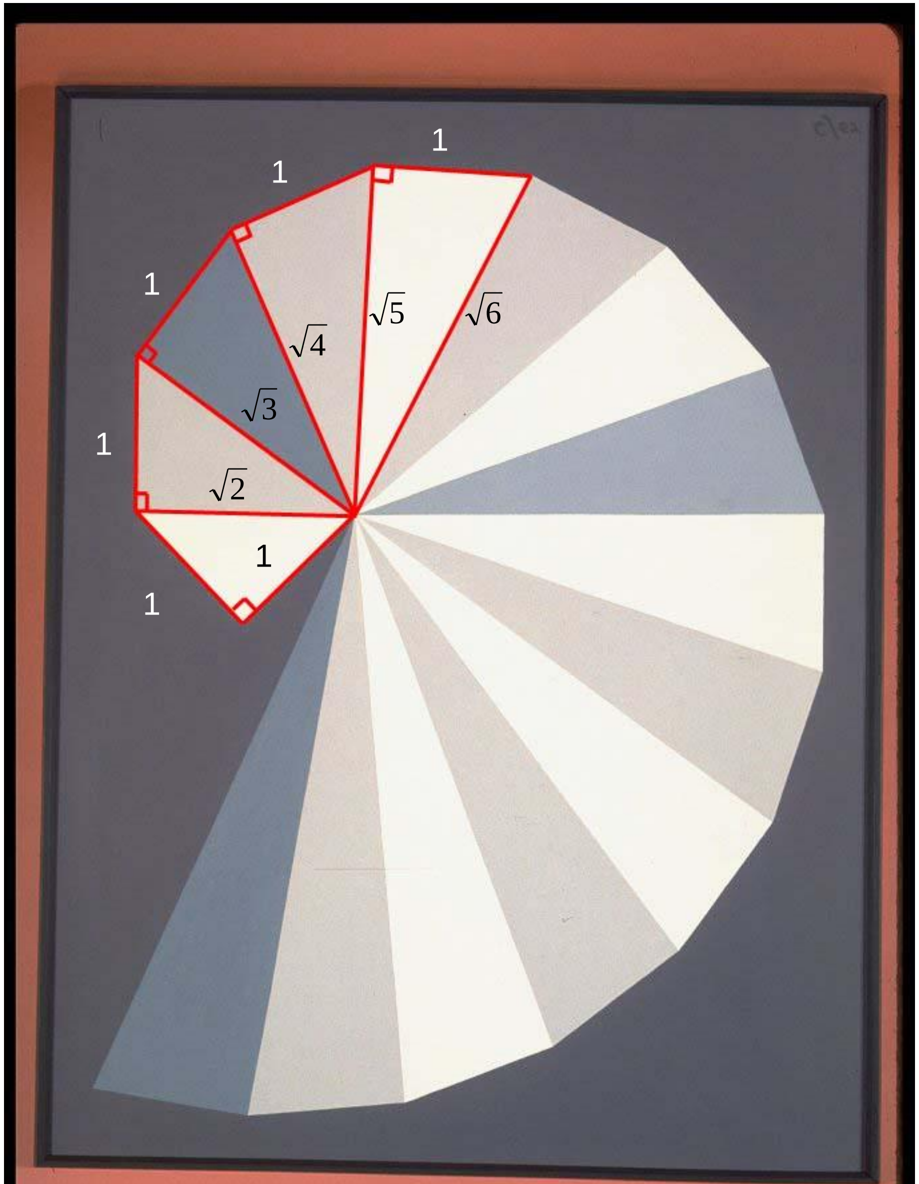

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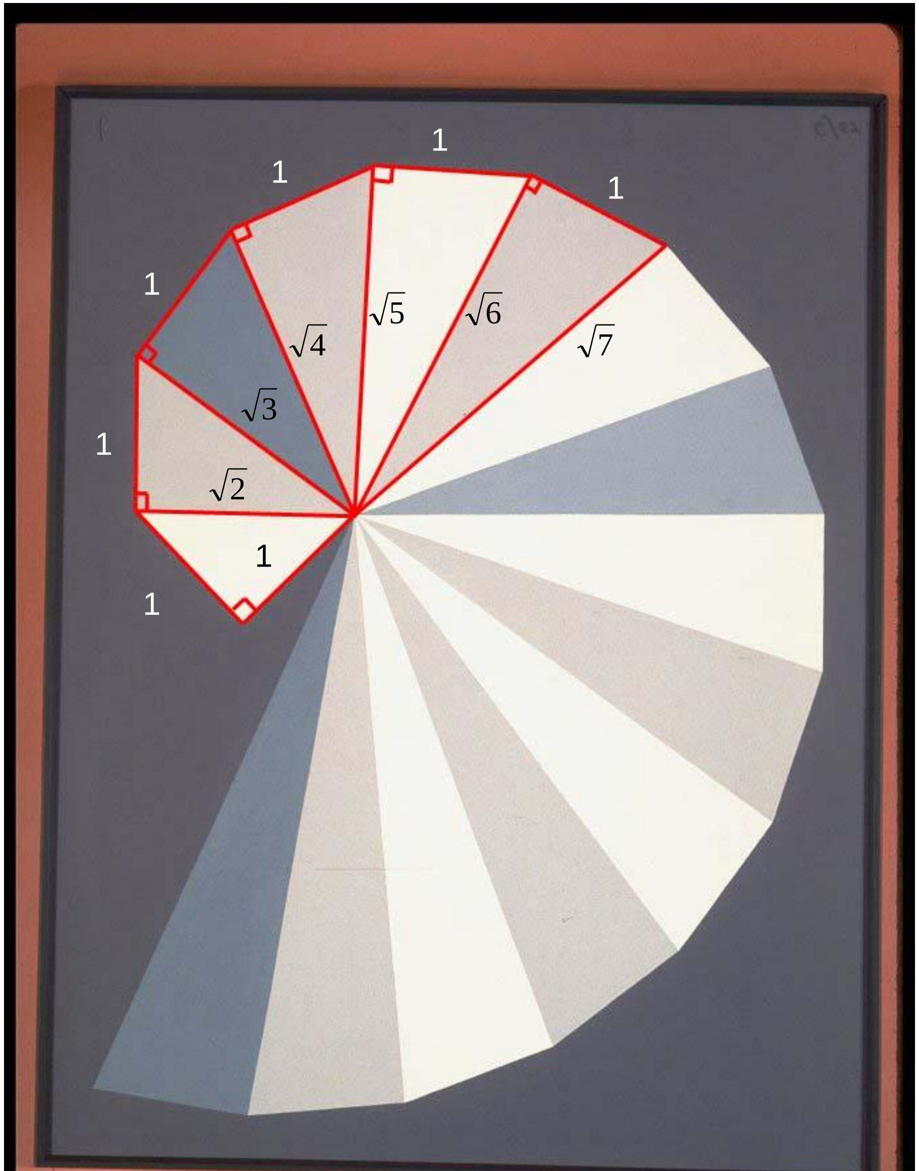

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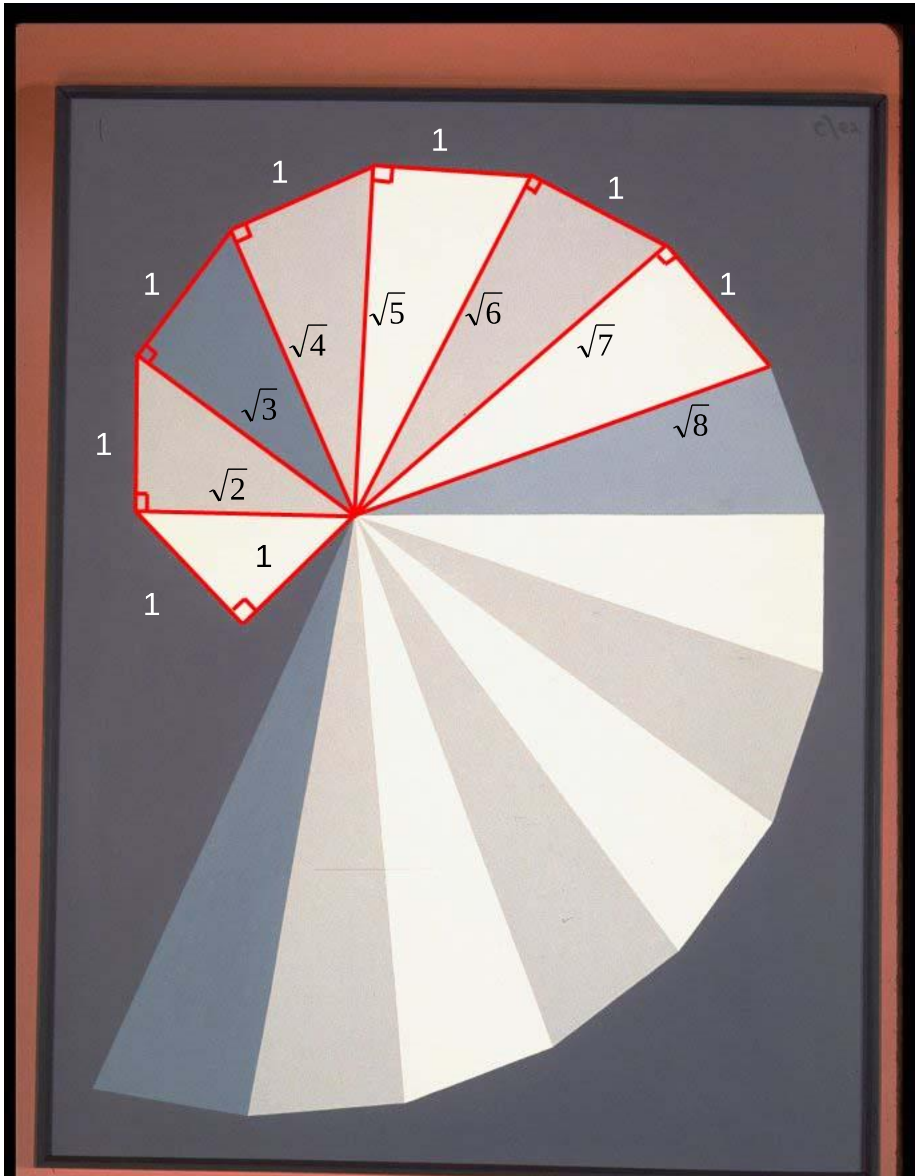

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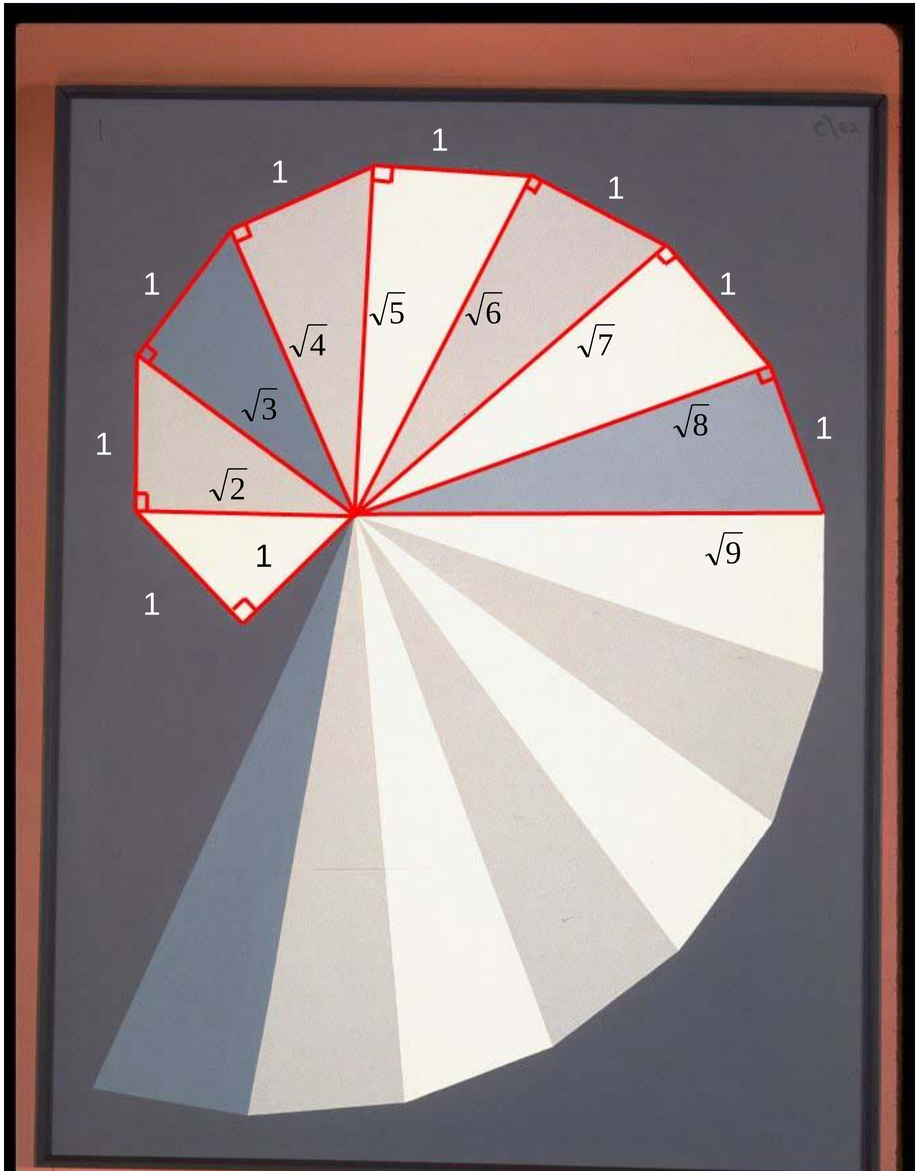

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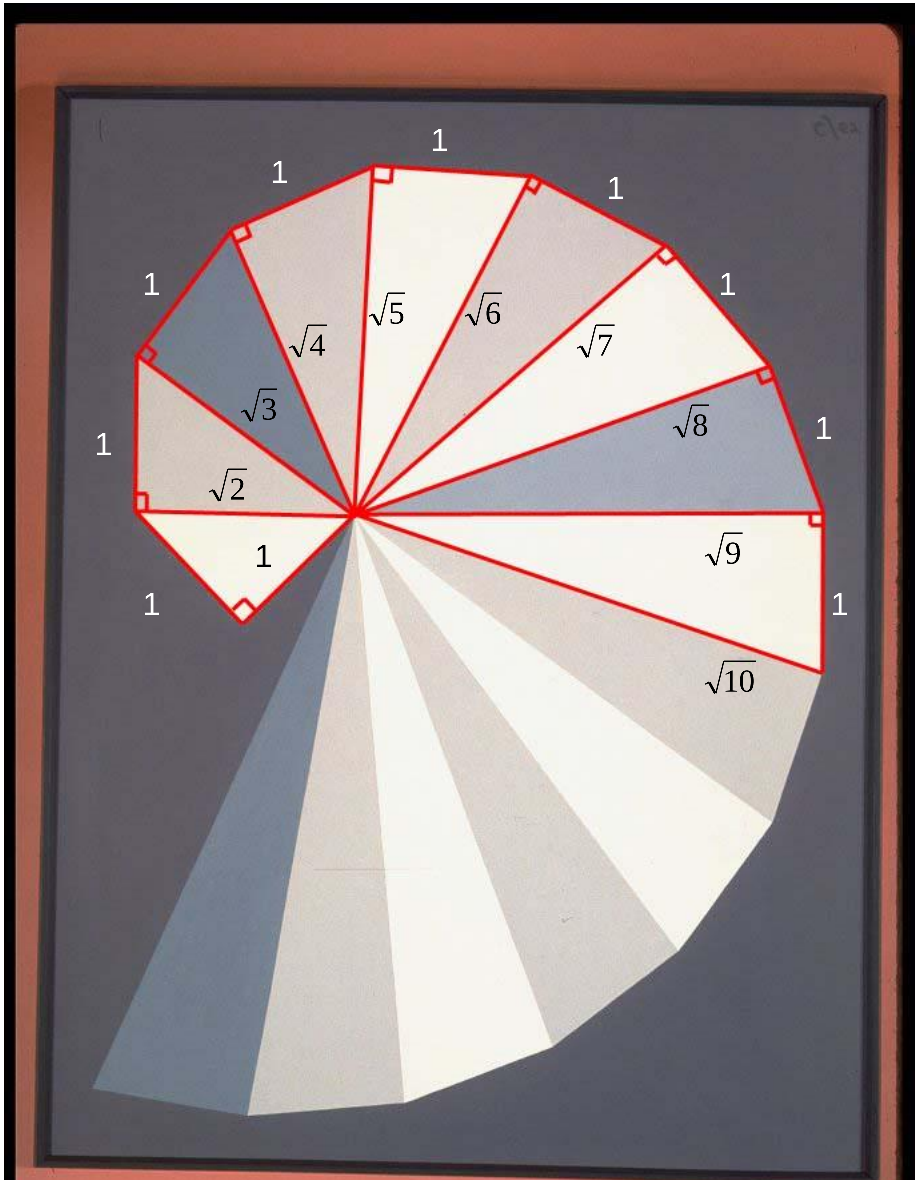

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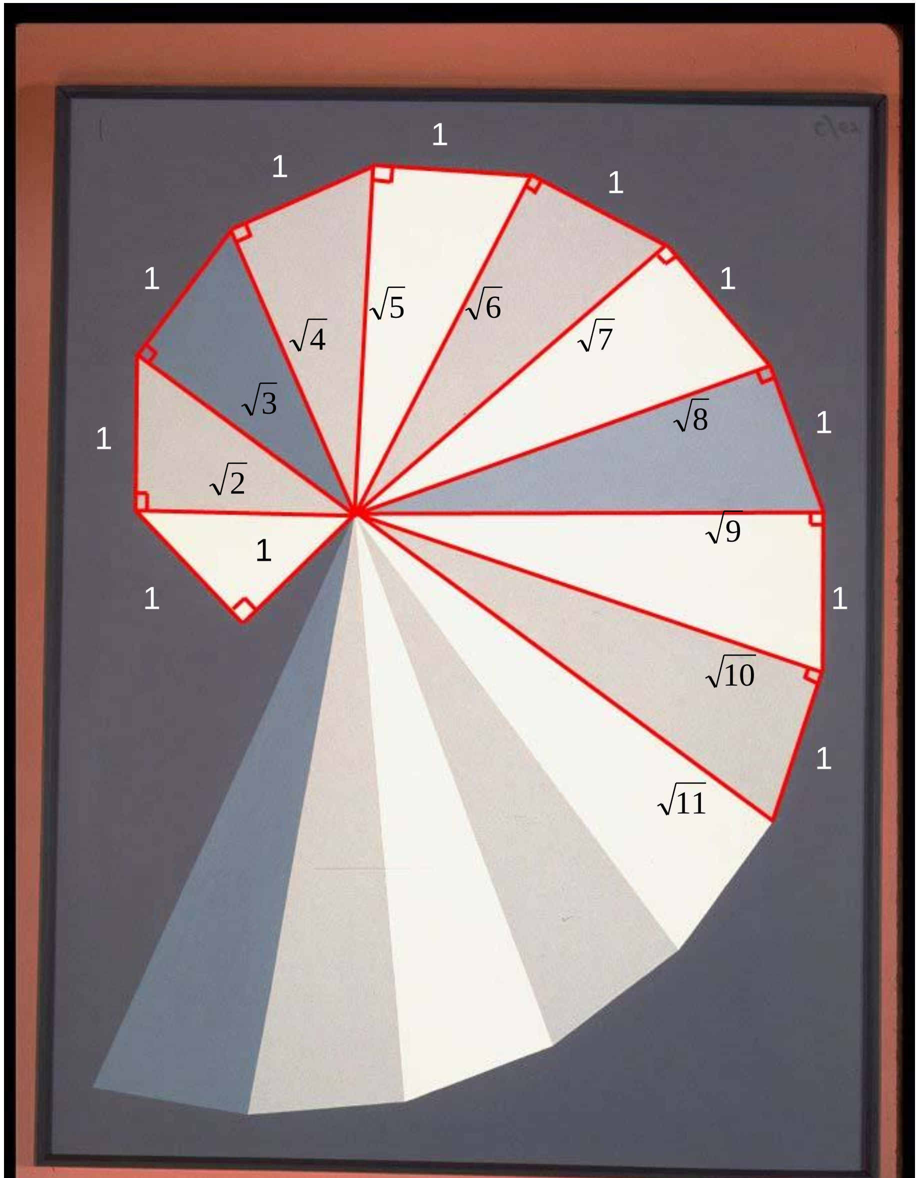

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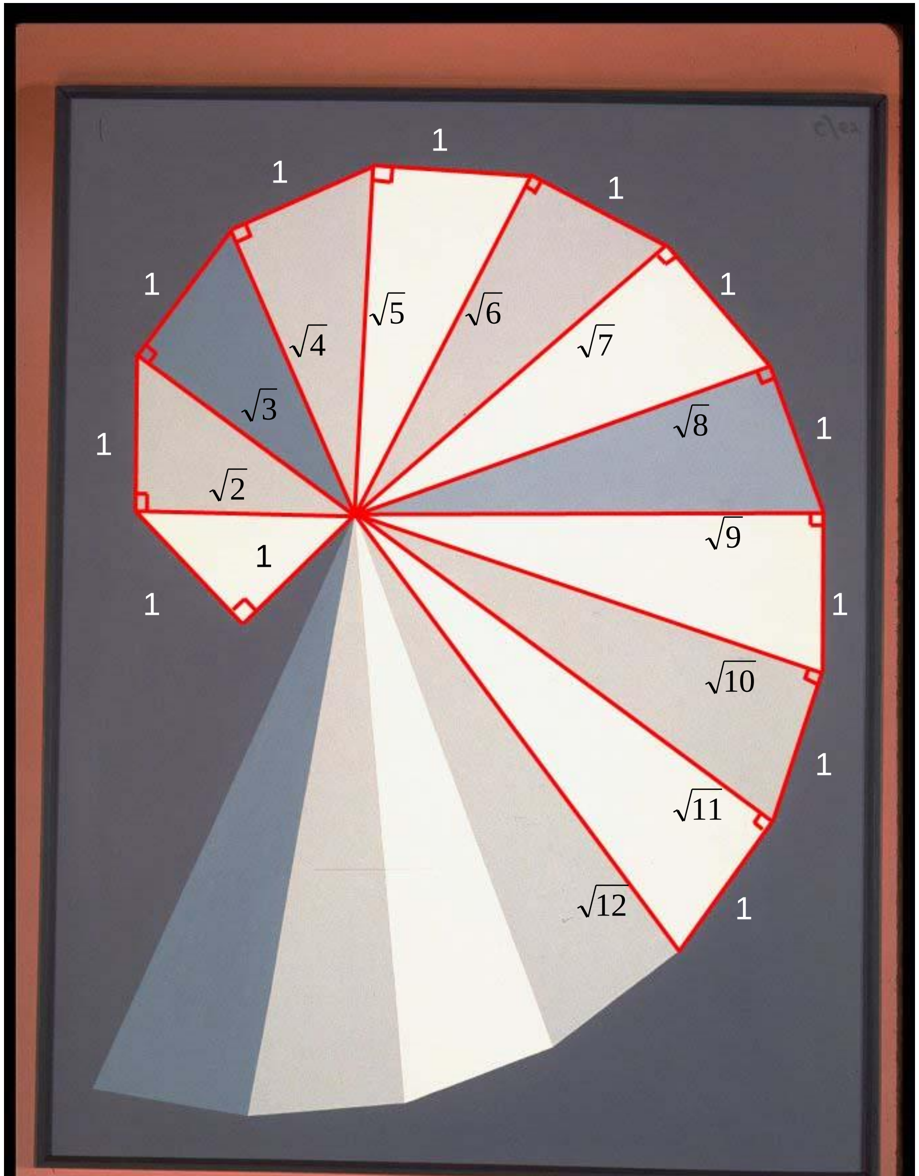

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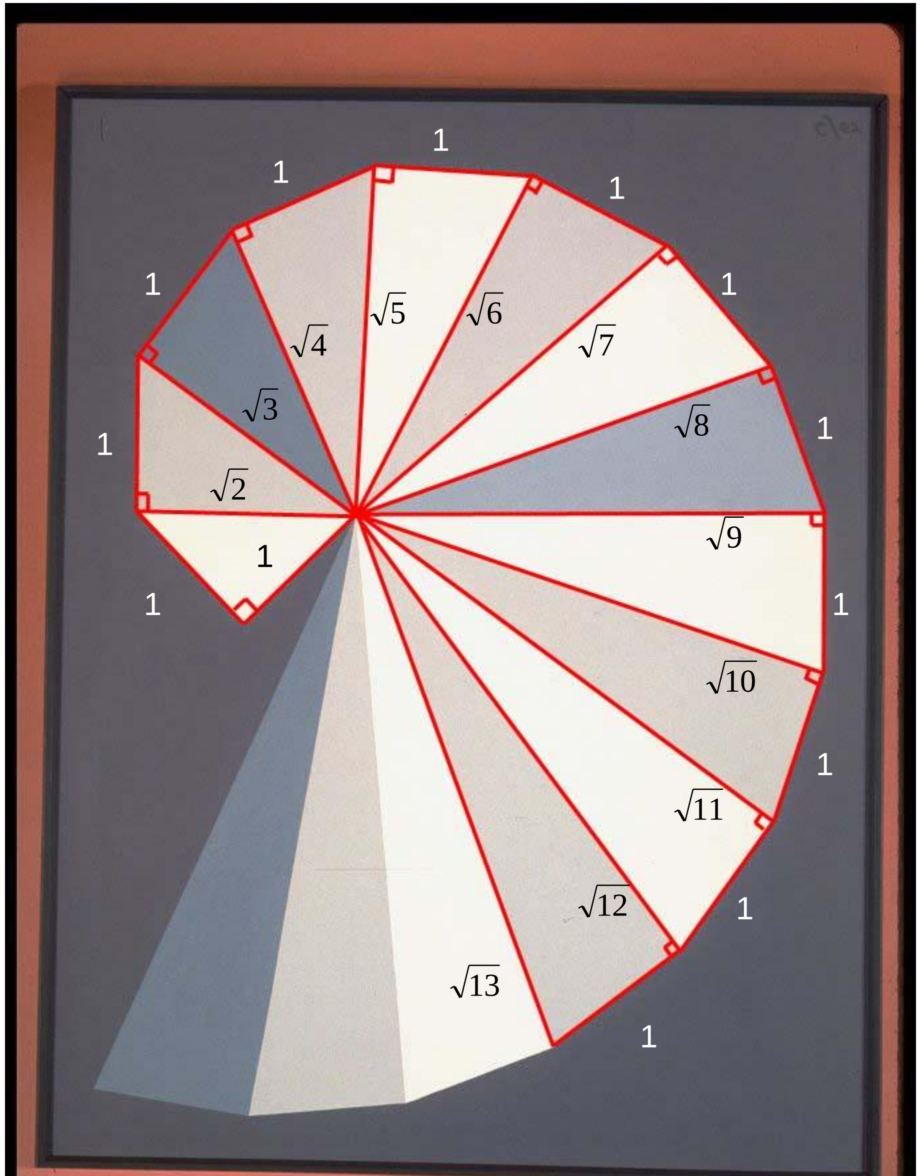

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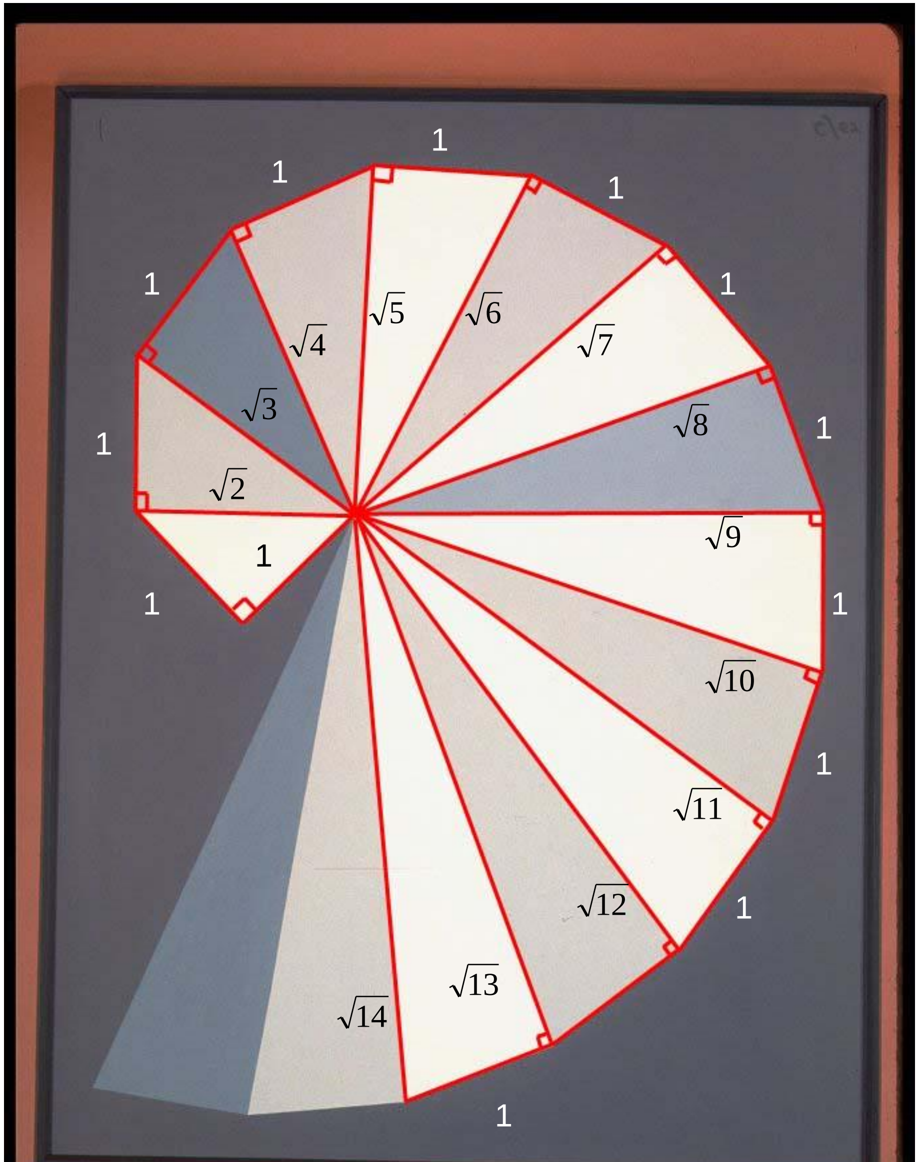

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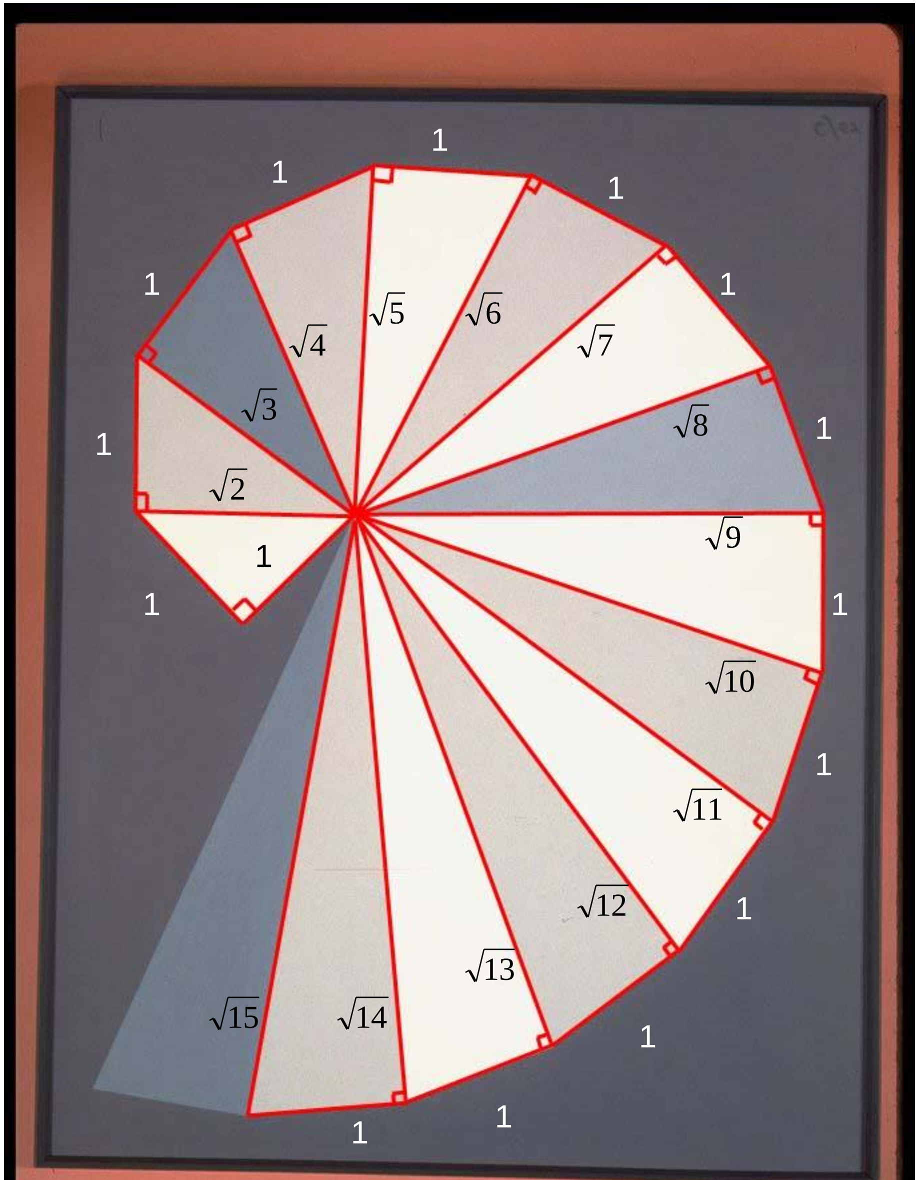

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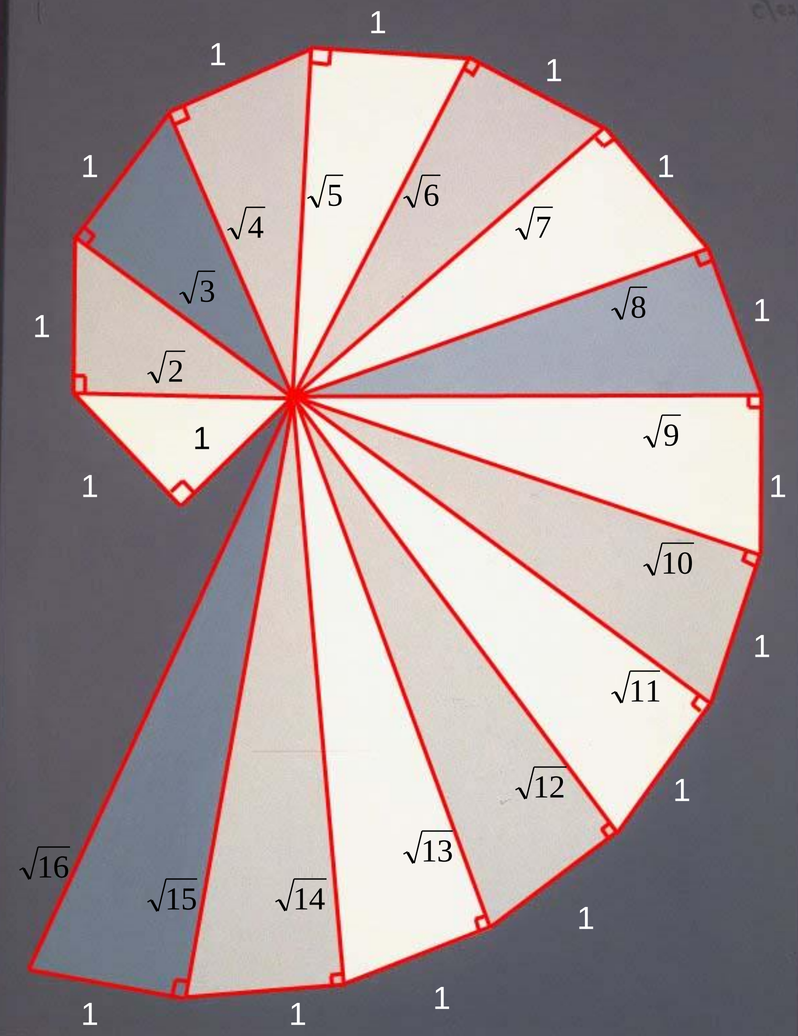
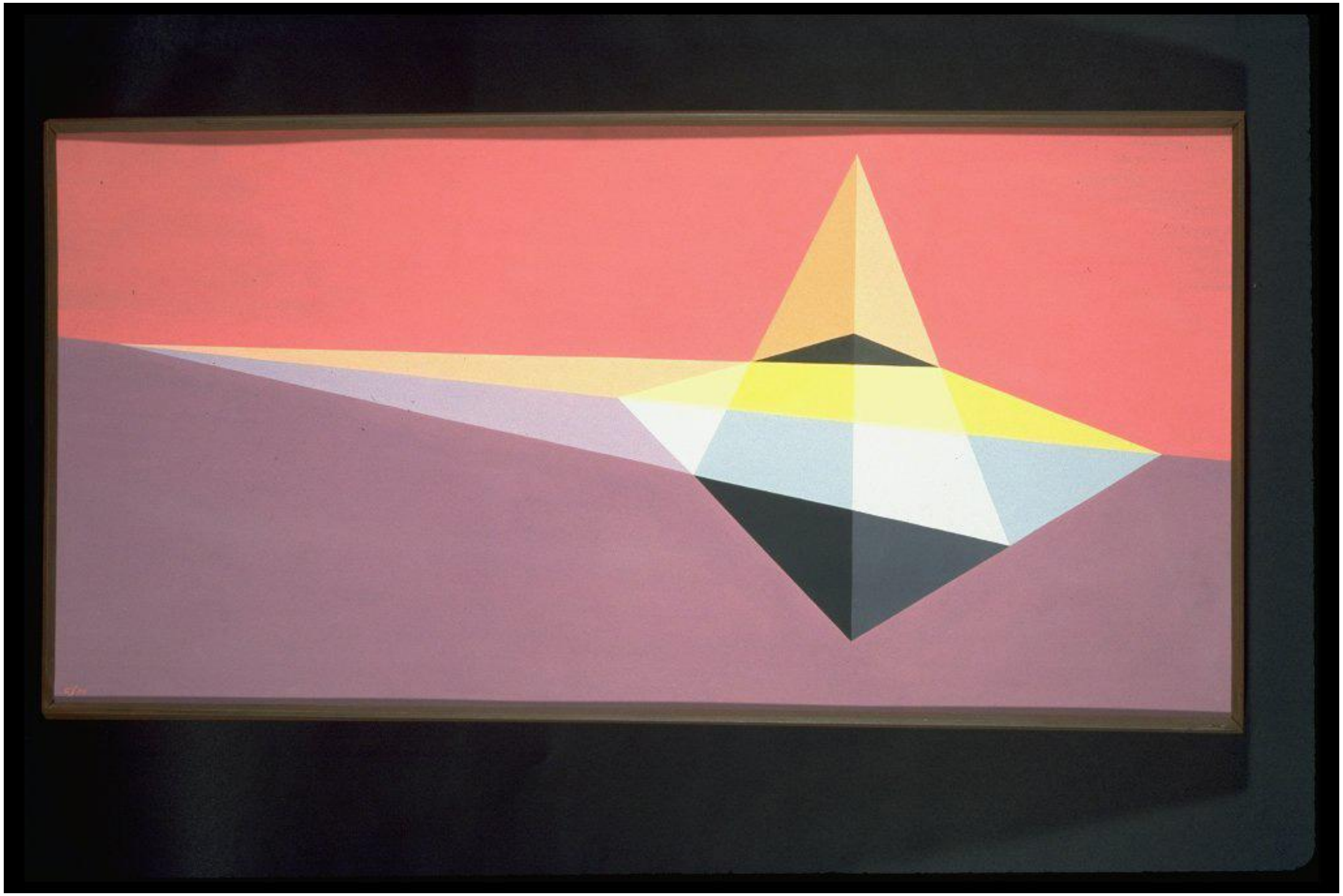


R. Calinger, *A Contextual History of Mathematics.*
Prentice Hall, 1999, p. 100.

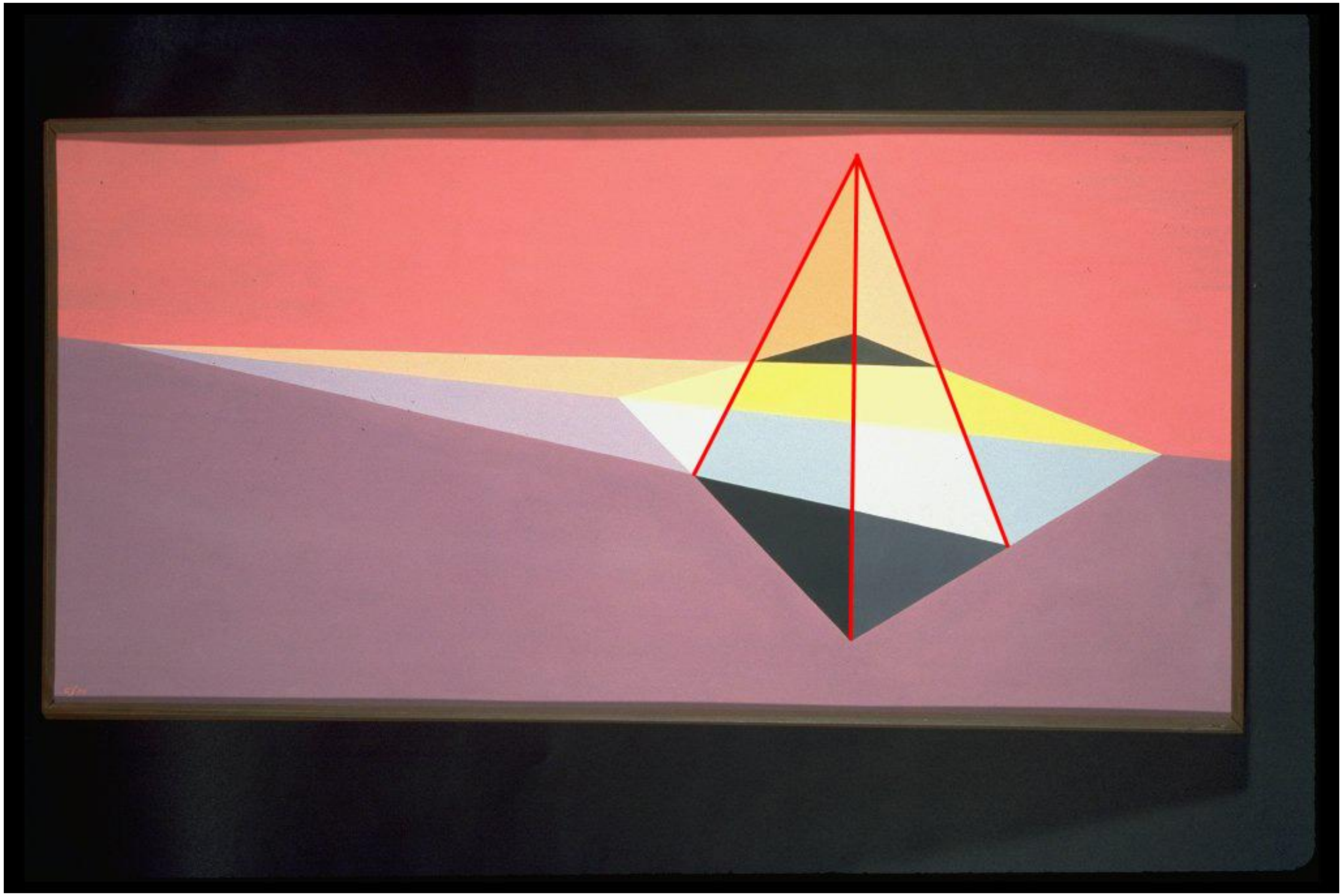
Desargues Theorem

If the corresponding vertices of two triangles ABC and XYZ lie on concurrent lines, the corresponding sides, if they intersect, meet in collinear points.

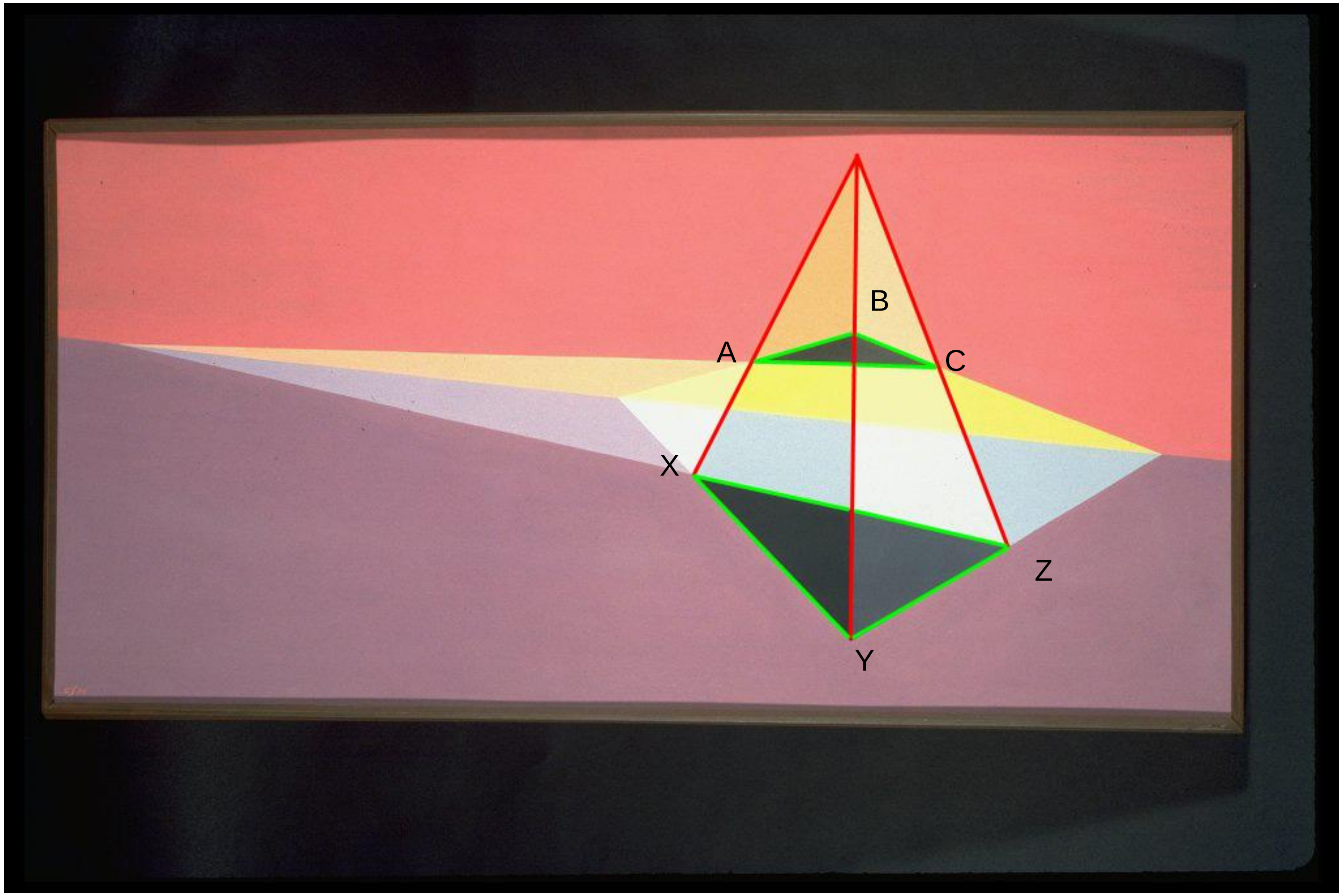
Desargues Theorem



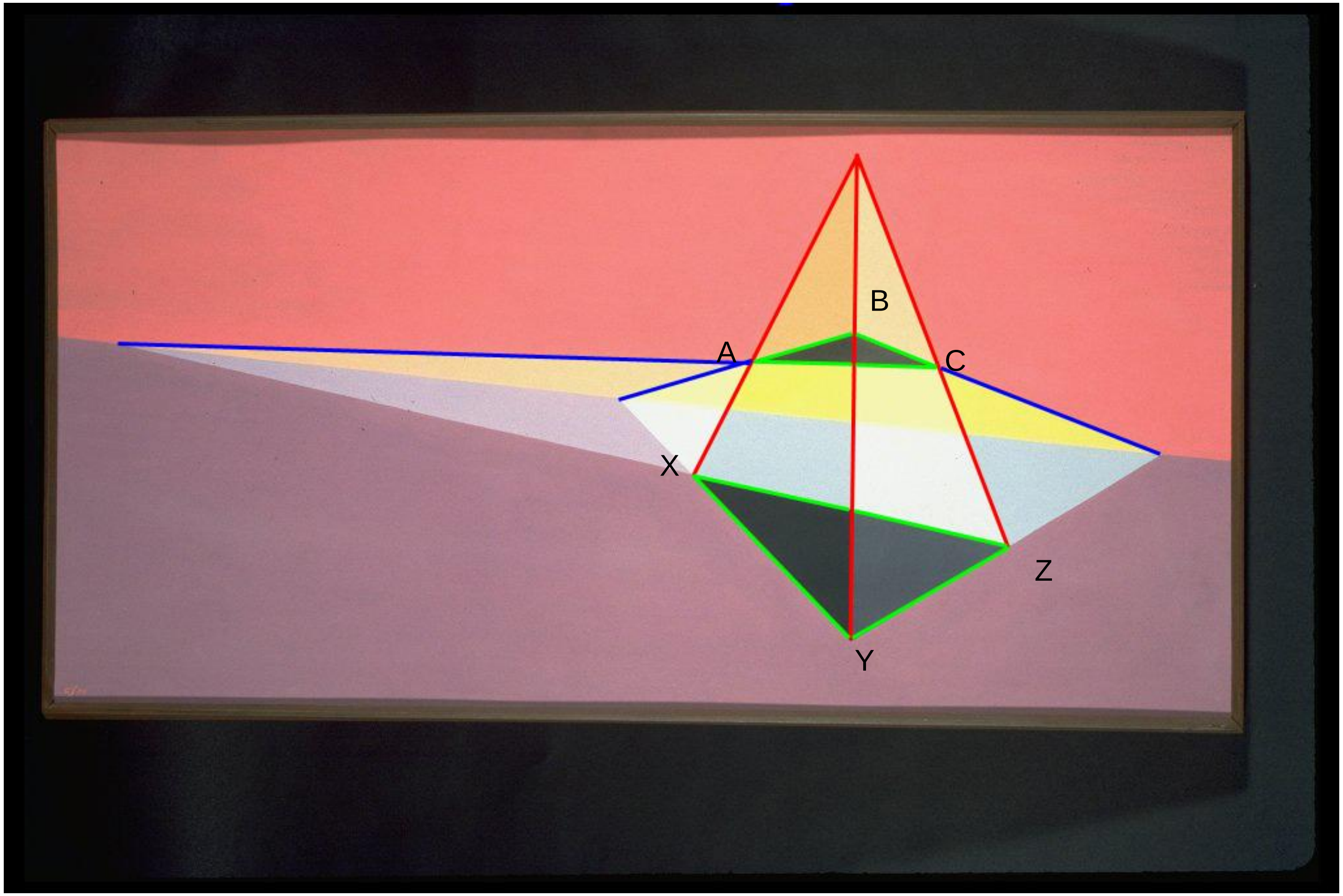
Desargues Theorem



Desargues Theorem

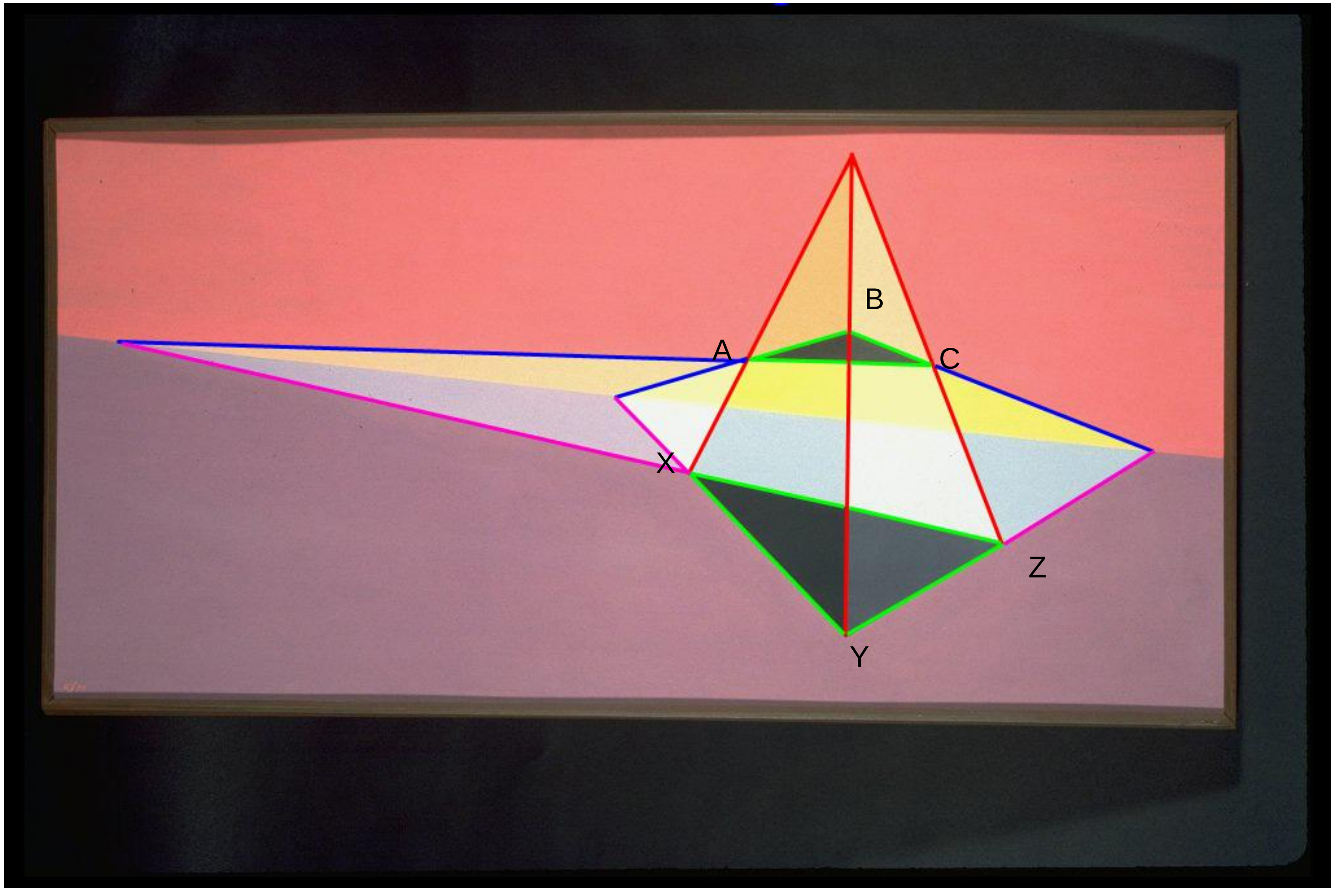


Desargues Theorem



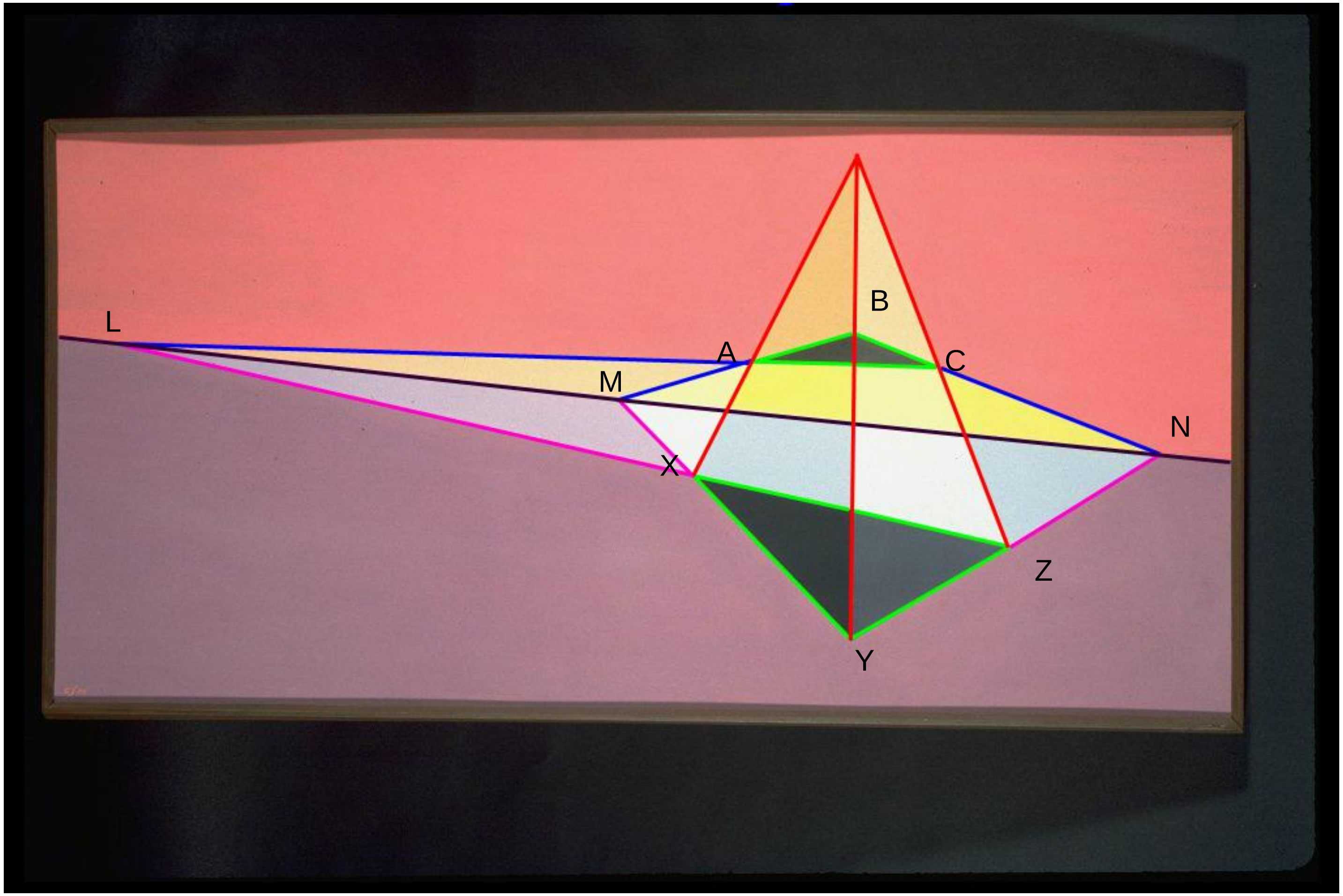
Extend the sides of triangle ABC.

Desargues Theorem



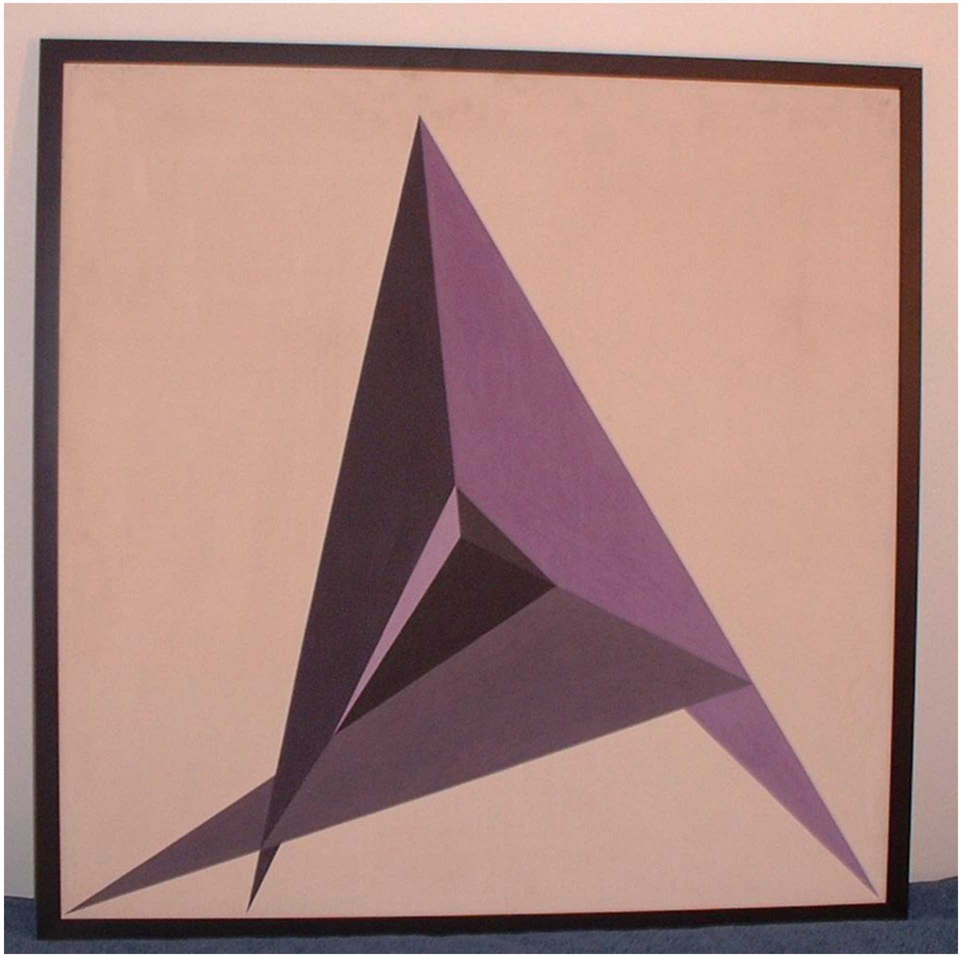
Extend the sides of triangle XYZ.

Desargues Theorem

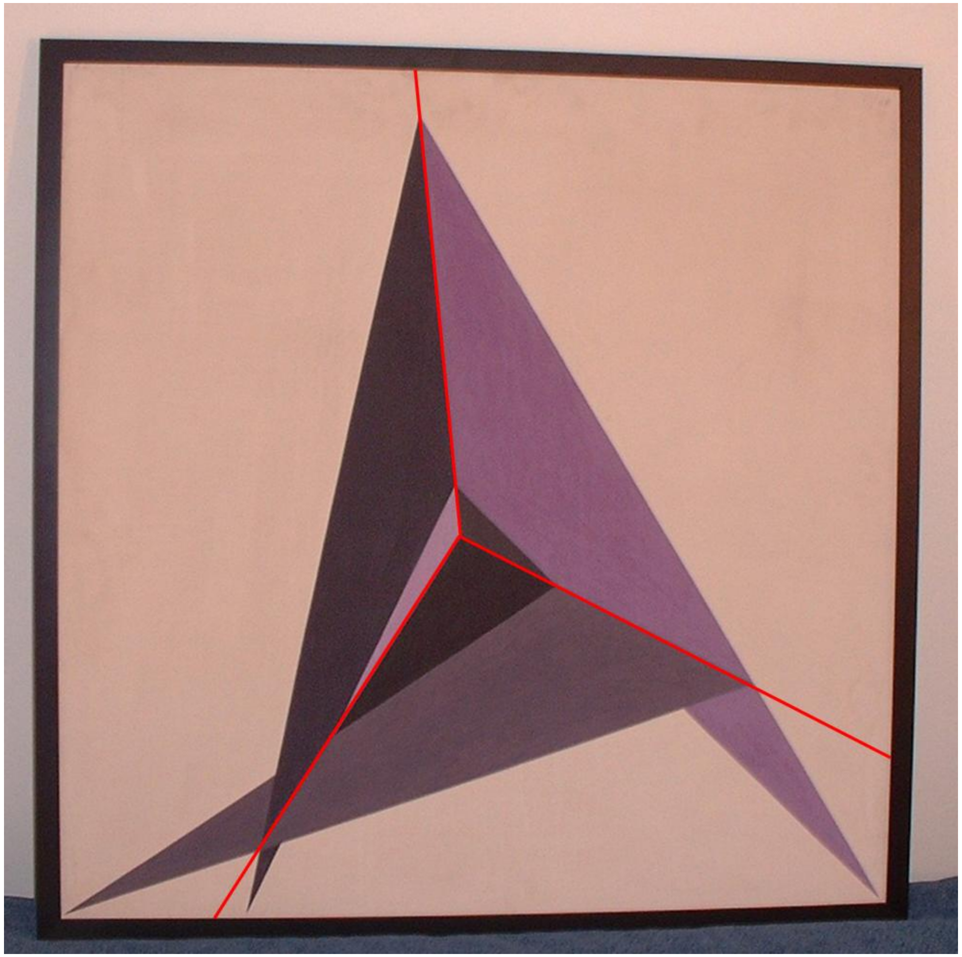


L, M, N are collinear.

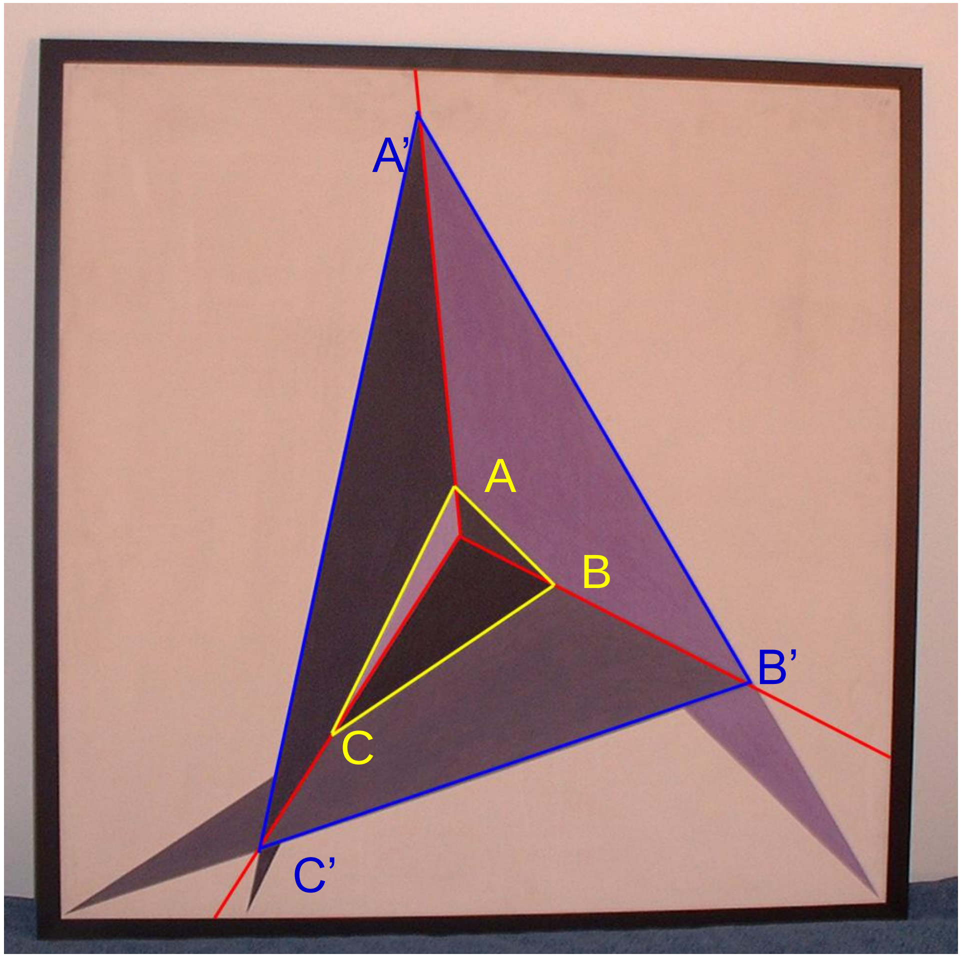
Desargues Theorem 2



Desargues Theorem 2

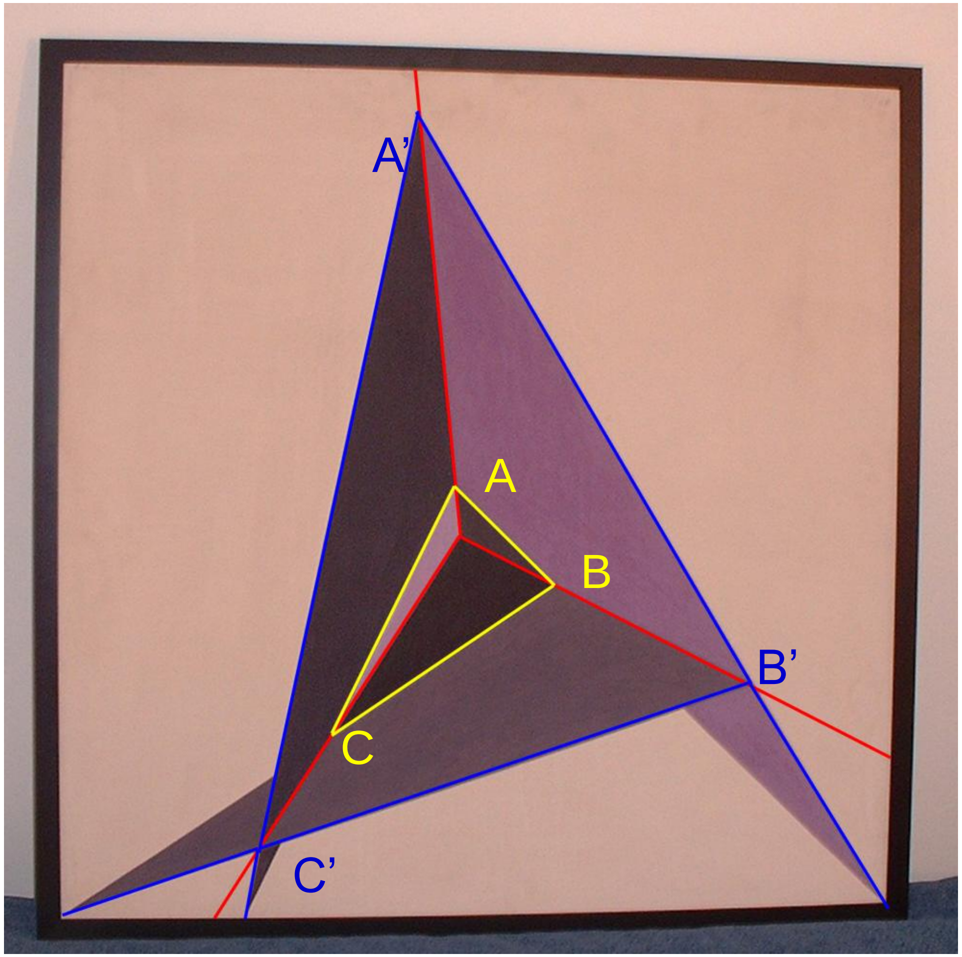


Desargues Theorem 2



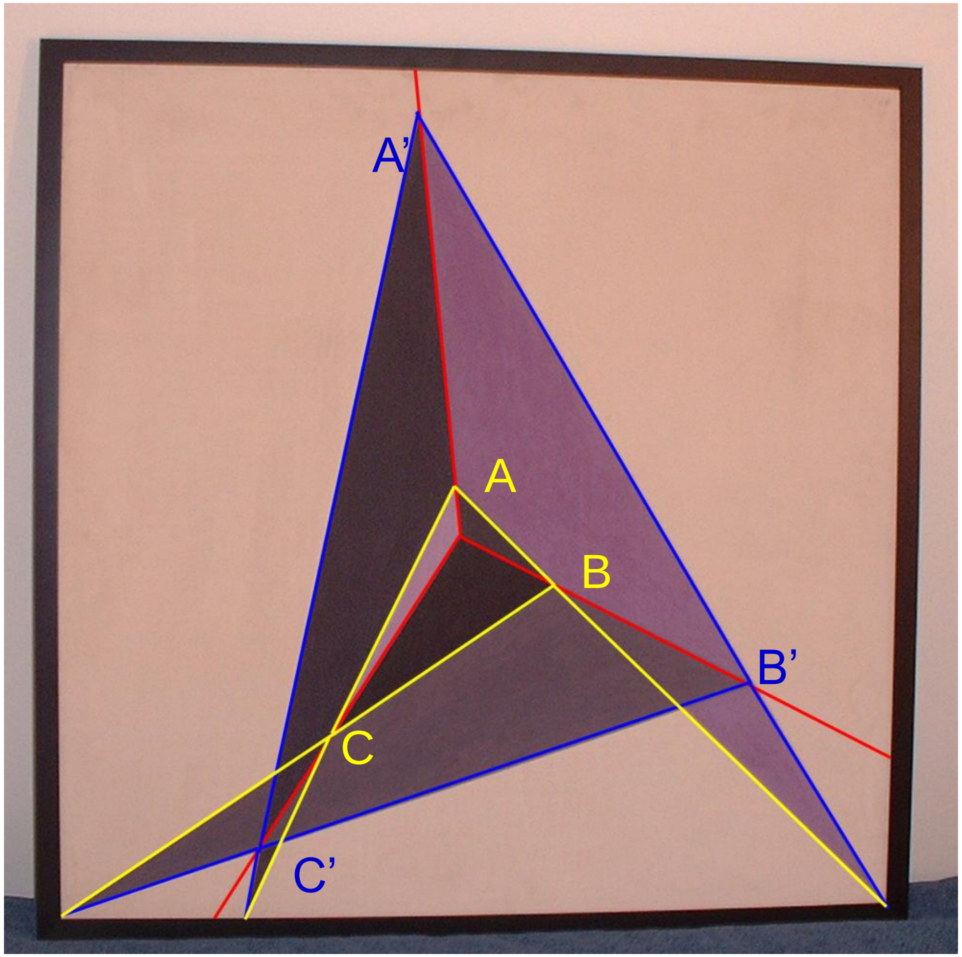
Desargues Theorem 2

Extend the sides of triangle A'B'C'



Desargues Theorem 2

Extend the sides of triangle ABC



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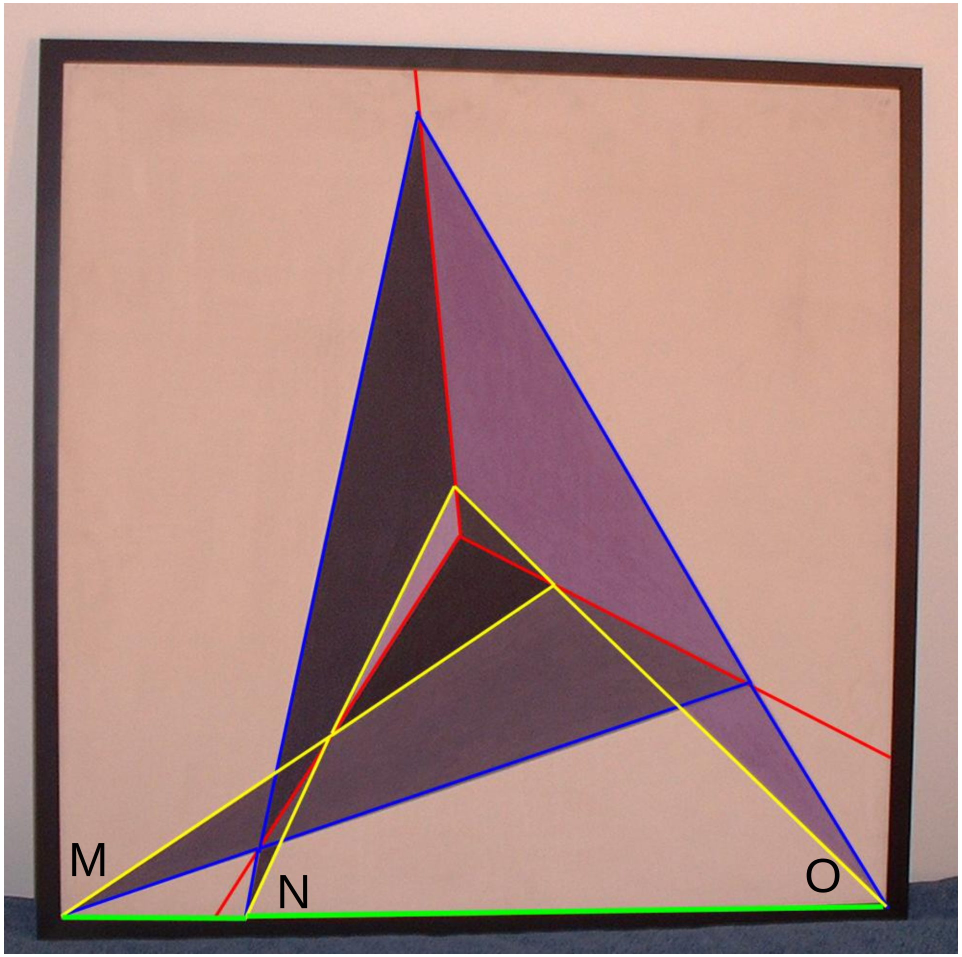


M, N, and O are collinear points.

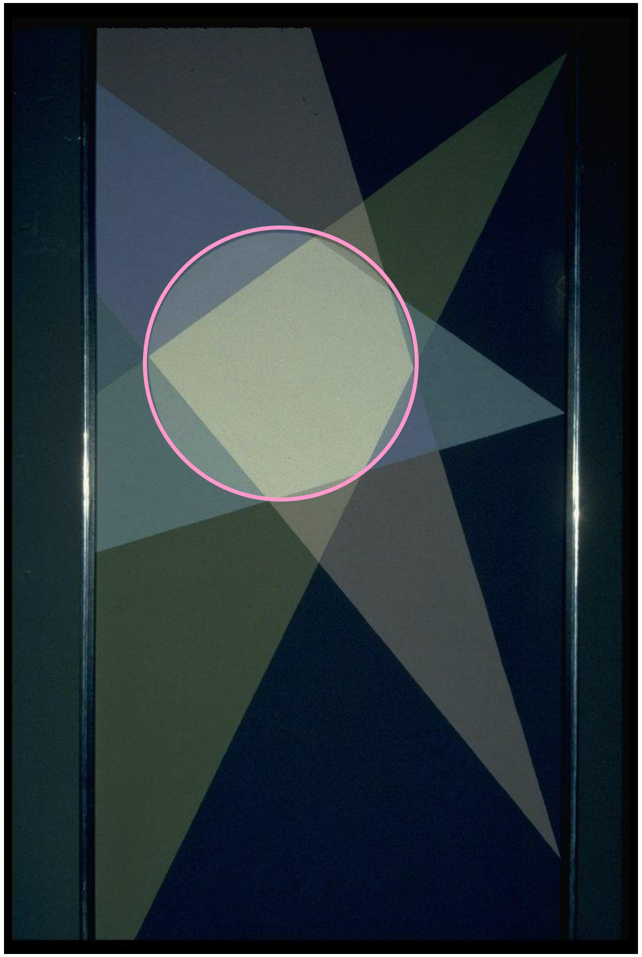
“Mystic” Hexagon

Pascal’s Hexagon Theorem

The three points of intersection of the opposite sides of a hexagon inscribed in a conic section lie on a straight line.

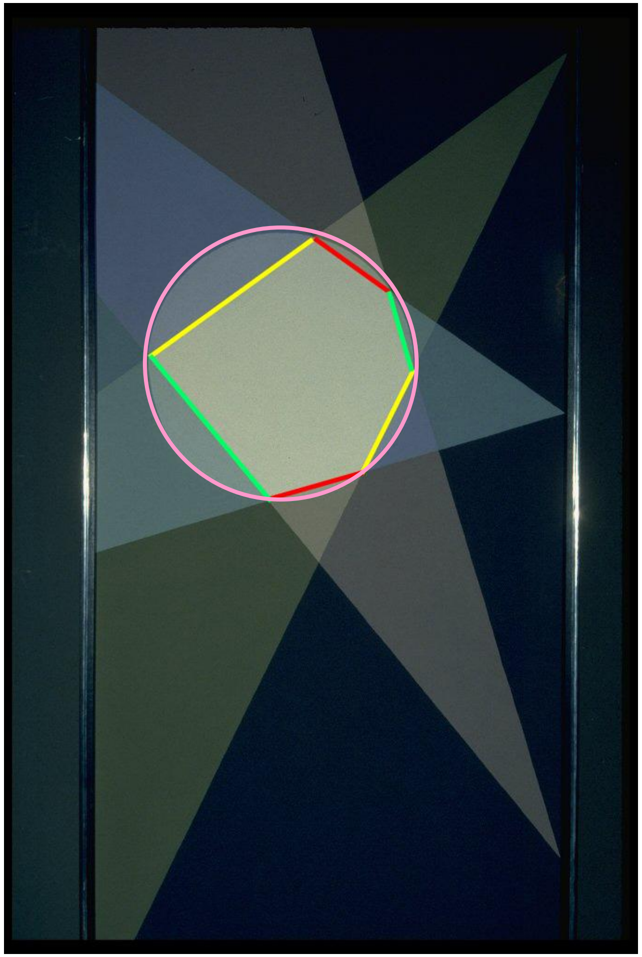
“Mystic” Hexagon

"Mystic" Hexagon



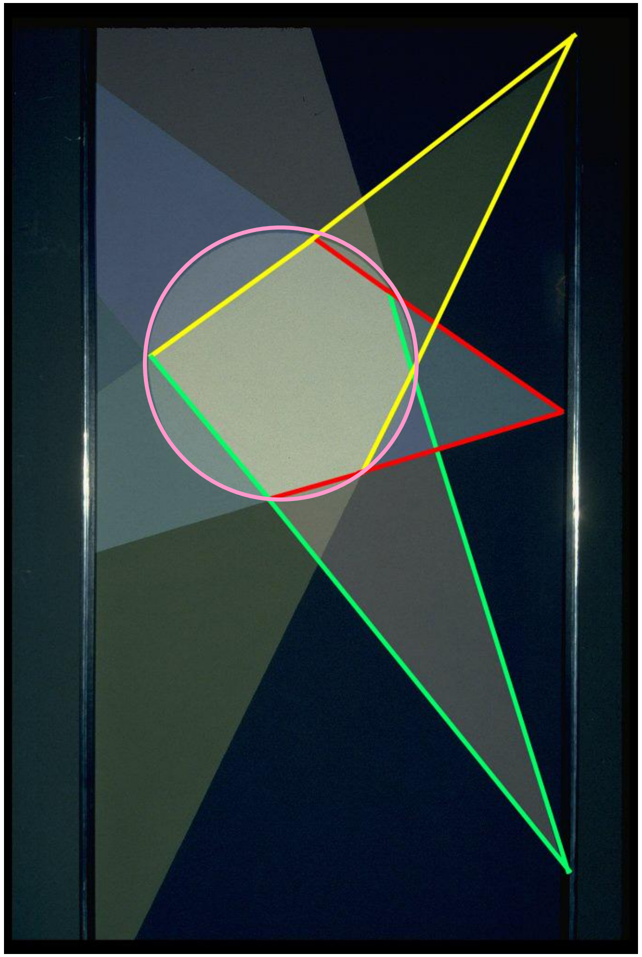
Begin with a circle

“Mystic” Hexagon



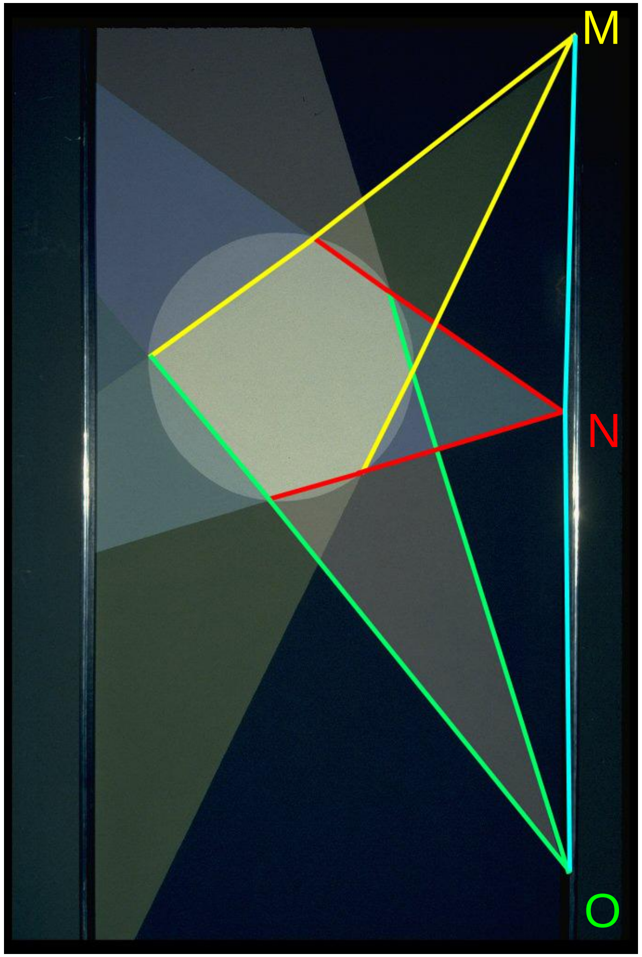
Inscribe a hexagon

“Mystic” Hexagon



Extend the sides.

“Mystic” Hexagon



M,N,O are collinear.

Classical Greek Problems

- Problems date back to the time of the Pythagoreans, c. 530 B.C.E.
- Construction done with a straight edge and a compass.
- Resolution of theorems occurred in the nineteenth or twentieth century, with modern application of algebra and number theory.

Classical Greek Problems

1. Trisection of an arbitrary angle
2. Quadrature of a circle
3. Duplication of a cube
4. Quadrature of a lune
5. Construction of a regular polygon

Quadrature of the Circle

$$x^2 = \pi \cdot R^2$$ **Let R = 1**

$$x^2 = \pi$$

$$x = \sqrt{\pi}$$

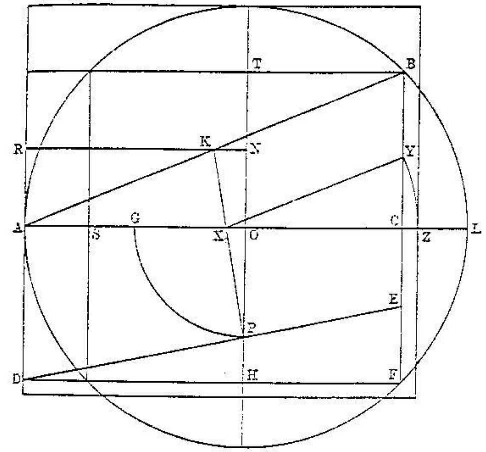
$\sqrt{\pi}$ **is Transcendental**

A Geometrical Look at Pi

The Mathematical Gazette in 1970 published “A Geometrical Look at Pi.”

Johnson was very proud of this construction and used it in at least three paintings.

3250. A geometrical look at $\sqrt{\pi}$



$$SC = \sqrt{2}(1{\cdot}414214) = TH$$

$$AC = 1 + \frac{\sqrt{2}}{2}(1{\cdot}707107)$$

$$BC = \frac{\sqrt{2}}{2}(0{\cdot}707107) = TO = OC$$

$$AB = \sqrt{(2 + \sqrt{2})}(1{\cdot}847759)$$

$$KN = AO(1) - \frac{AC}{2} = \frac{2 - \sqrt{2}}{4}(0{\cdot}146447)$$

$$NP = \frac{1}{2} + \frac{\sqrt{2}}{4}(0{\cdot}853553)$$

$$\left(\frac{KN}{NP} = 0{\cdot}171573\right)$$

$$OP(0{\cdot}5) \cdot \frac{KN}{NP} = XO(0{\cdot}085787)$$

$$AO(1) - XO = AX(0{\cdot}914213)$$

$$OC + XO = XC(0{\cdot}792894)$$

$$XY \parallel AB\left(\frac{AB}{AC} = 1{\cdot}082392\right)$$

$$XC \cdot \frac{AB}{AC} = XY(0{\cdot}858222) = XZ$$

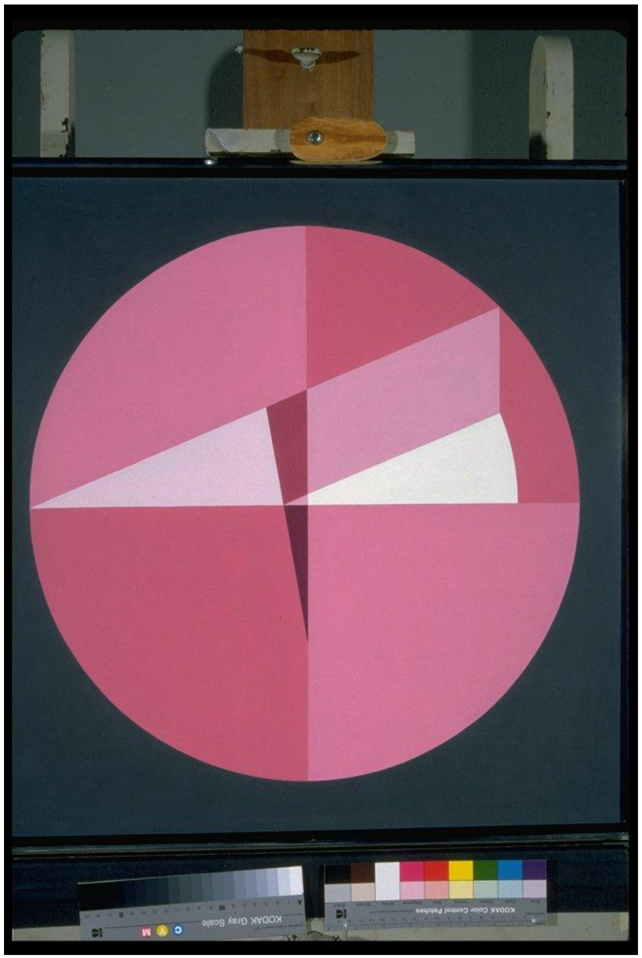
$$AX + XZ = AZ(1{\cdot}772435)$$

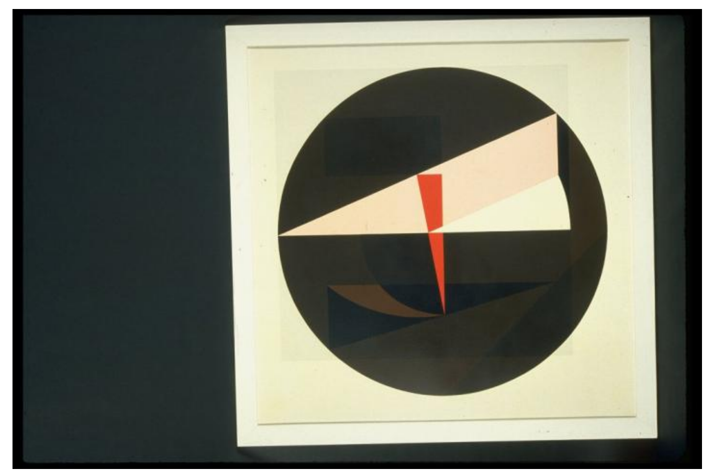
$$AZ = \sqrt{\pi} - 0{\cdot}00001$$

74 Rowayton Avenue,
Rowayton, Conn.

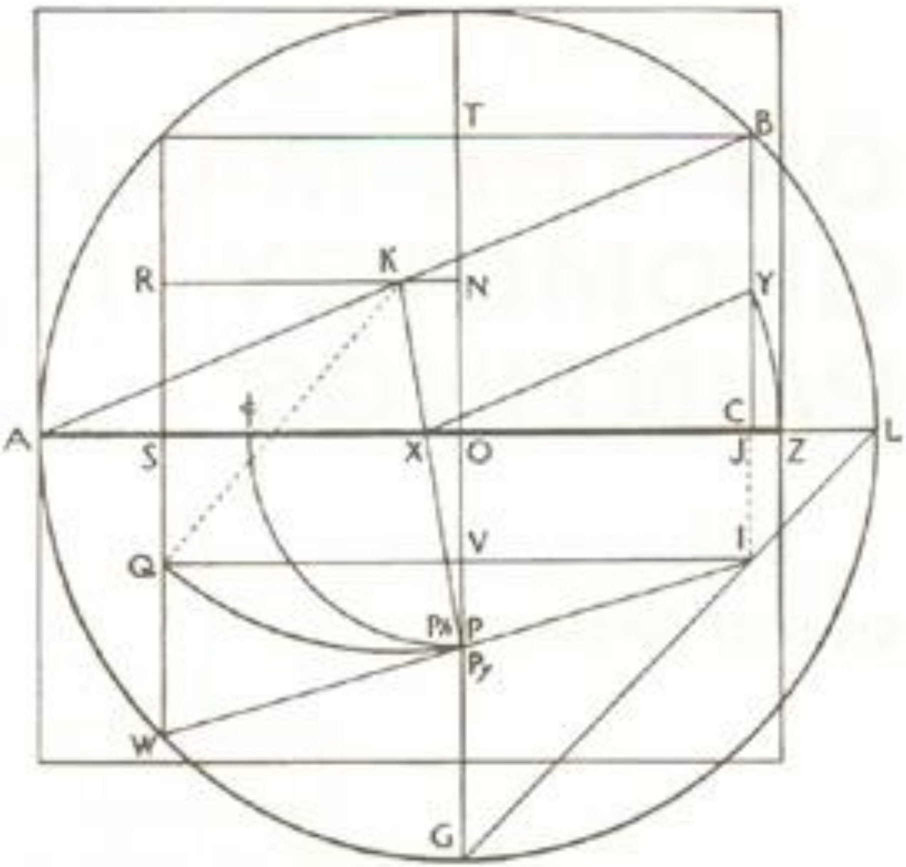
CROCKETT JOHNSON

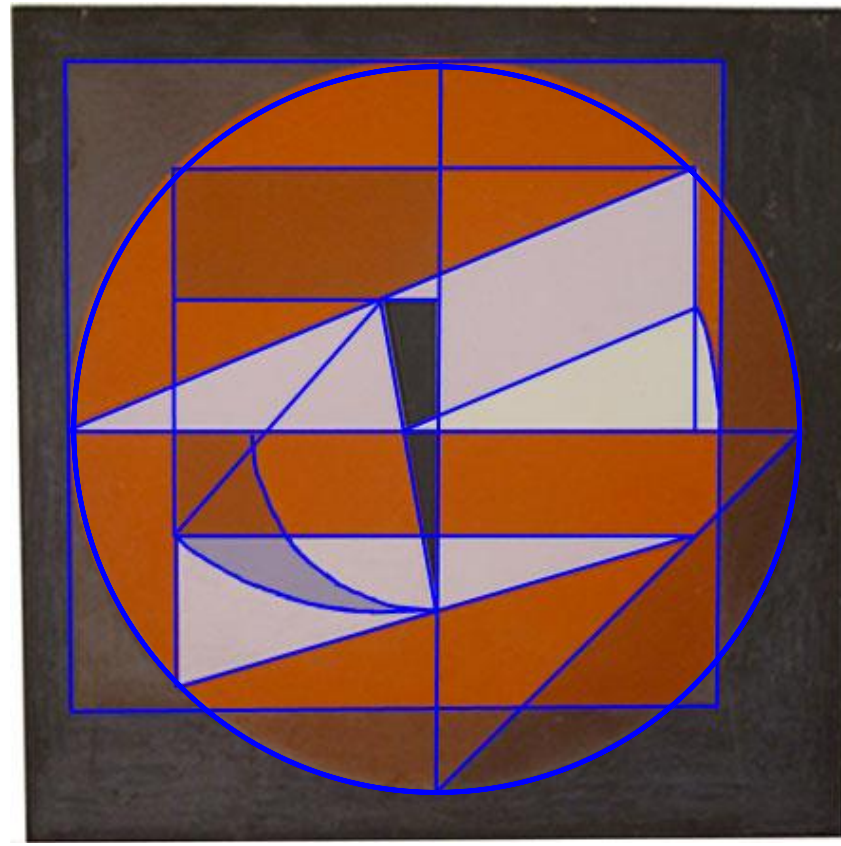
Square Root of Pi -.00001





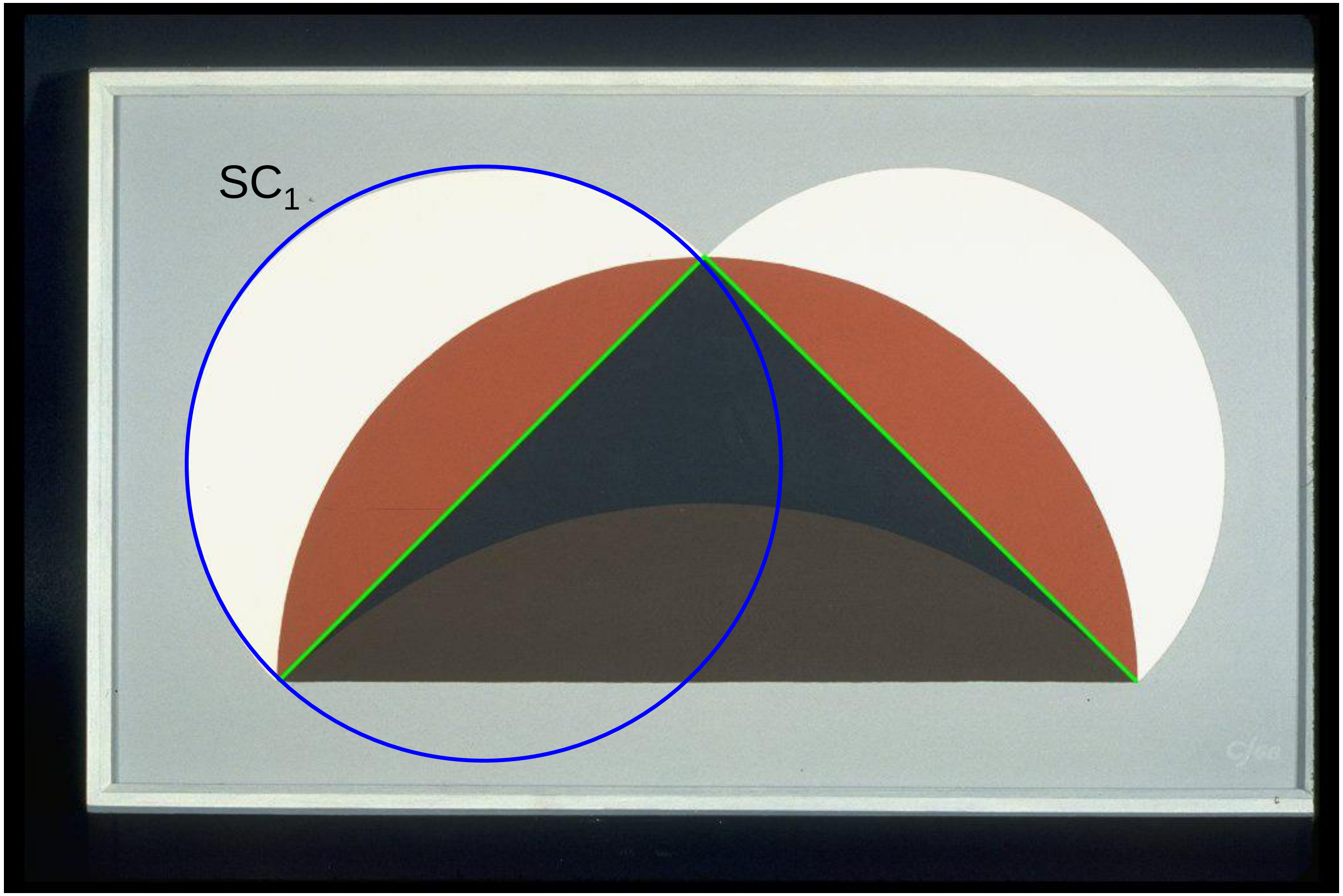
Square Root of Pi -.00001





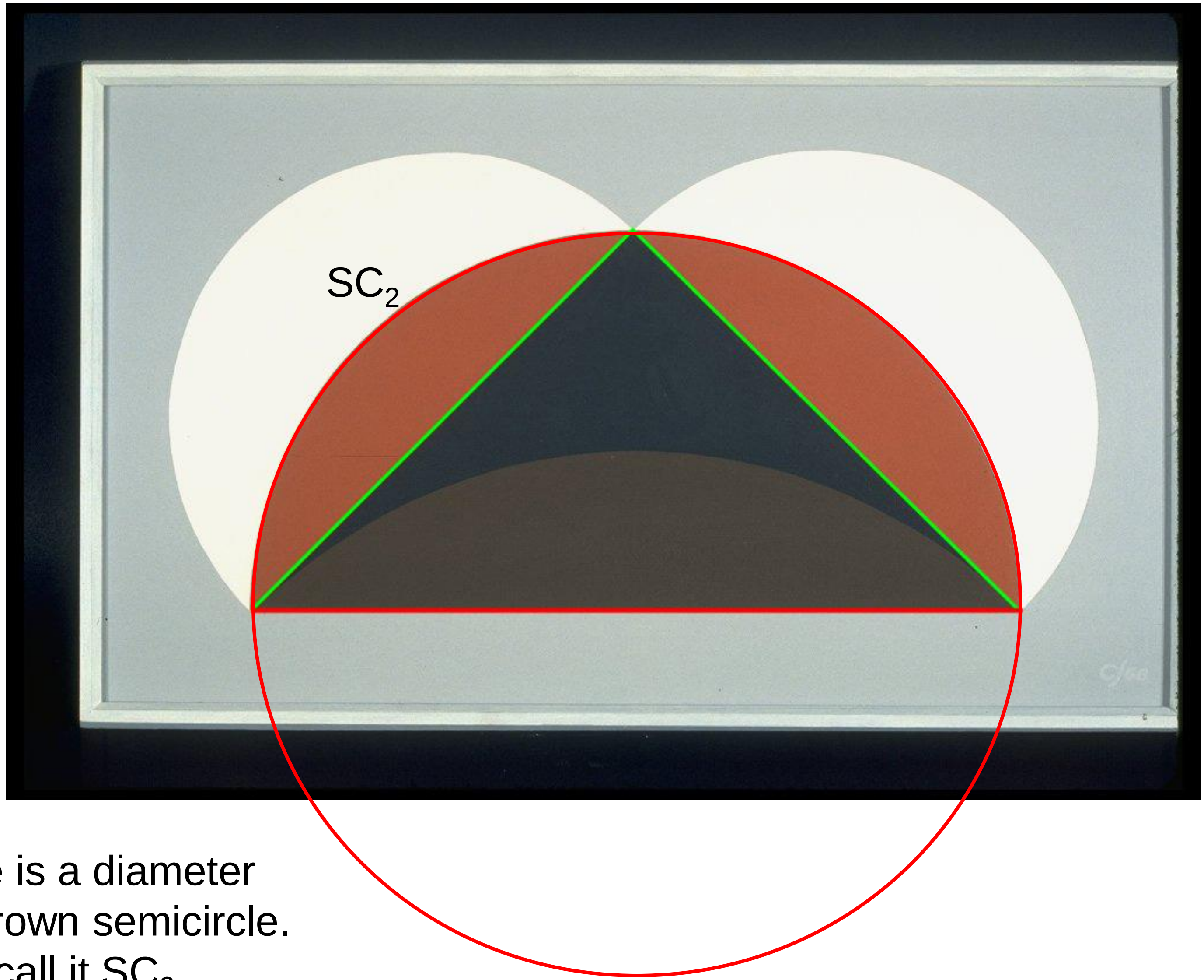
Quadrature of a Lune

Quadrature of a Lune



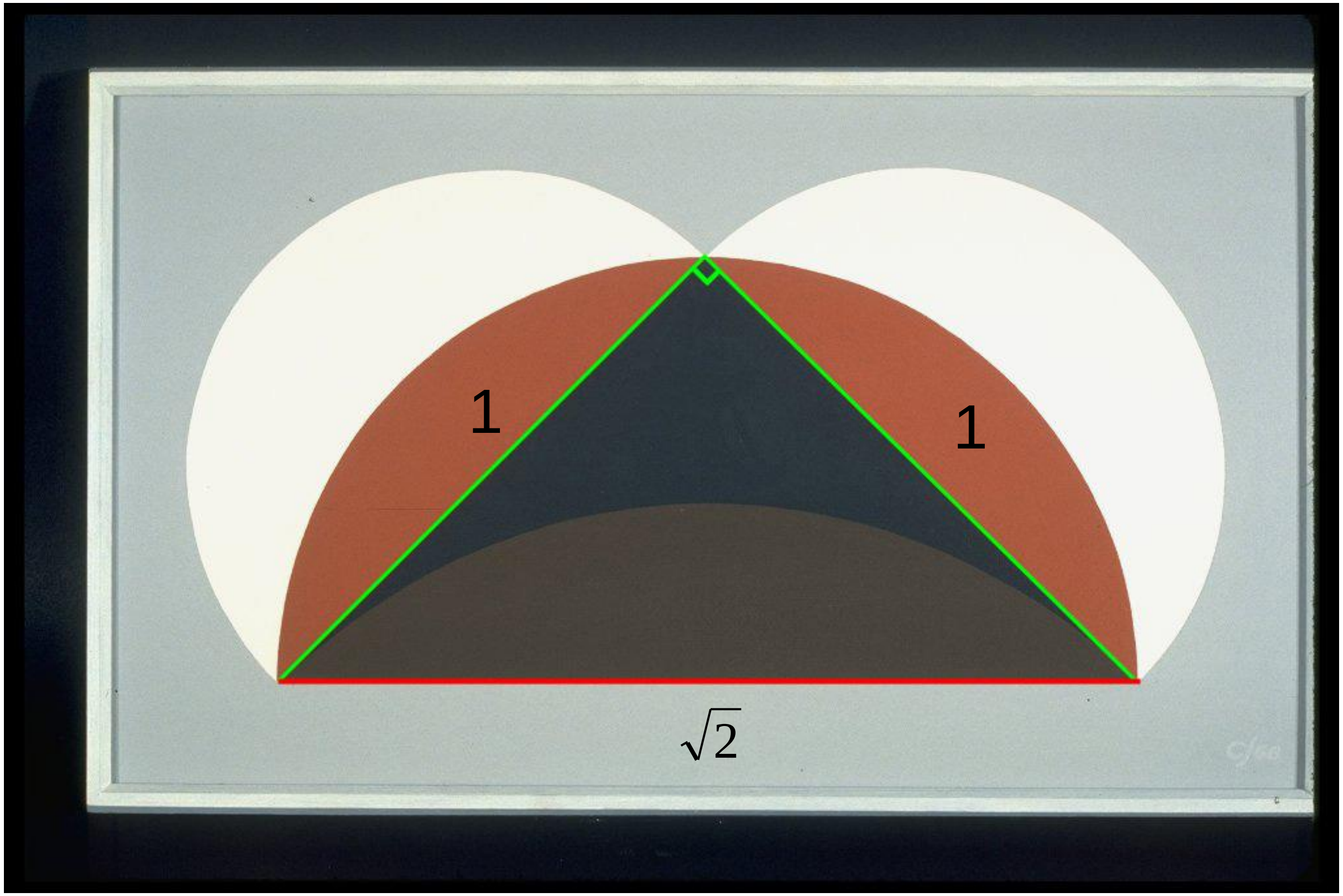
Green lines represent diameters to the semicircles.
We will call it SC_1.

Quadrature of a Lune



Red line is a diameter
to the brown semicircle.
We will call it SC_2.

Quadrature of a Lune



Some Proof:

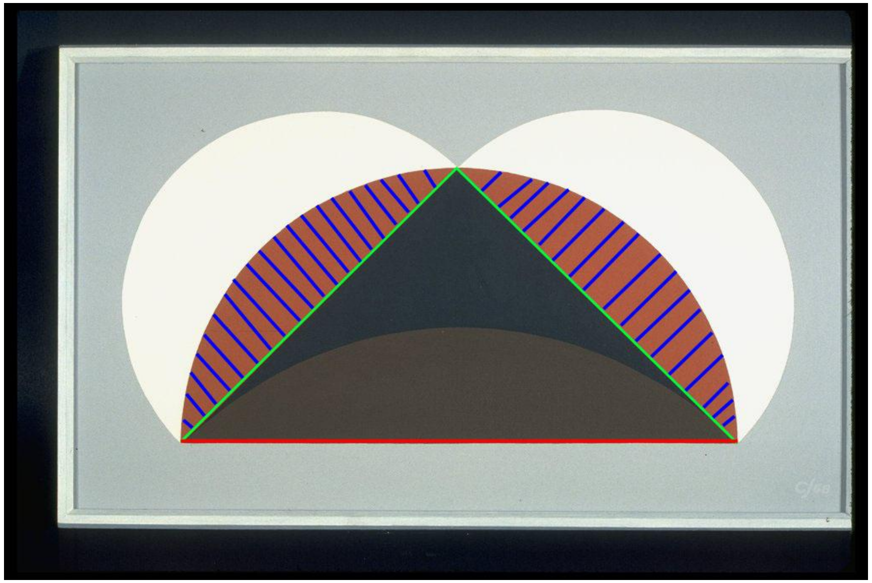
$$\frac{Area\ SC_1}{D_1^{\ 2}} = \frac{Area\ SC_2}{D_2^{\ 2}}$$

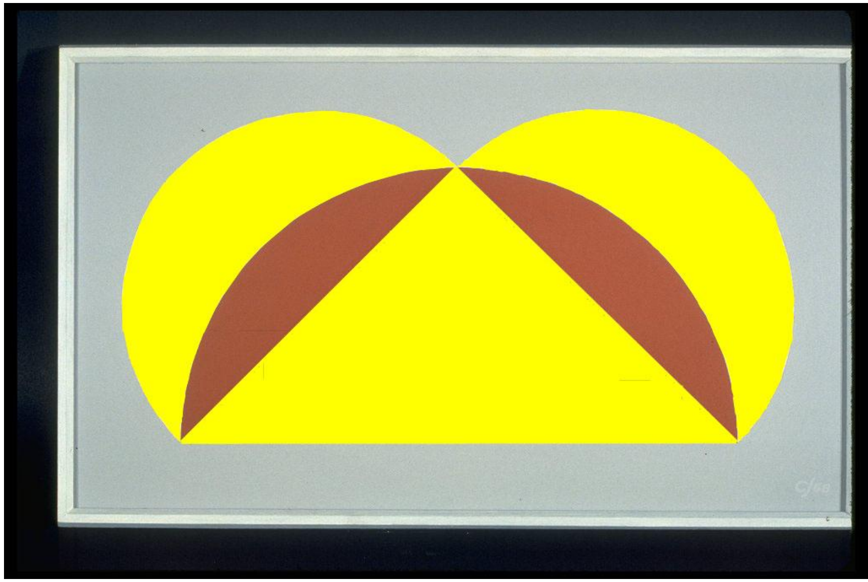
$$\Rightarrow \frac{Area\ SC_1}{1} = \frac{Area\ SC_2}{2}$$

$$\Rightarrow 2Area\ SC_1 = Area\ SC_2$$

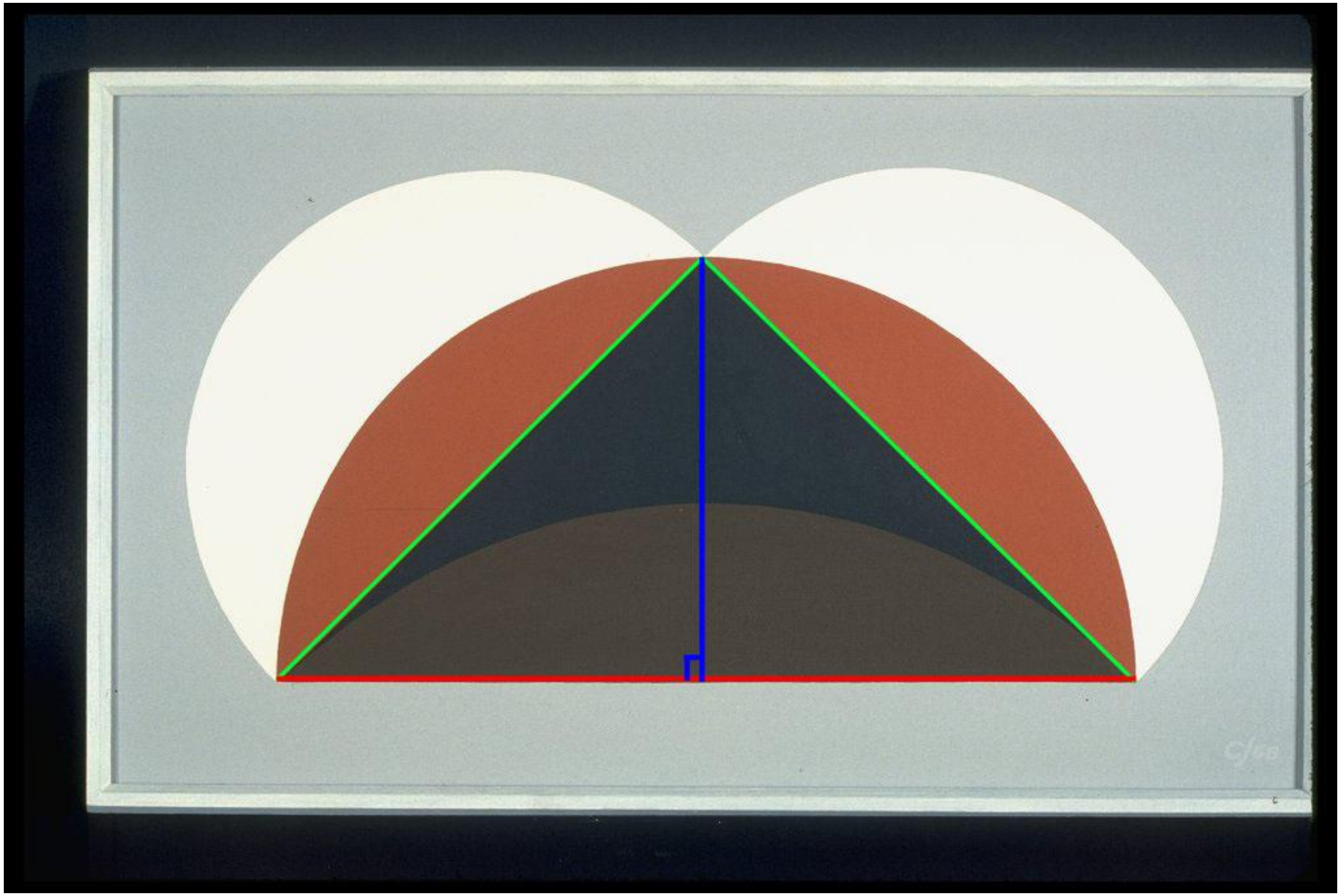
Therefore, the sum of the areas of the two smaller semicircles is equal in area to the larger semicircle.

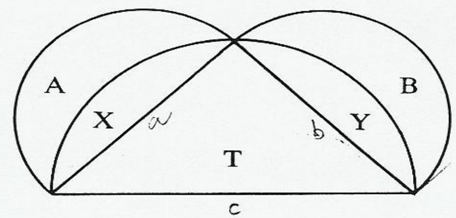
Quadrature of a Lune





Quadrature of a Lune



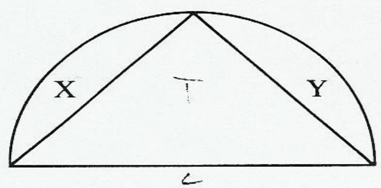


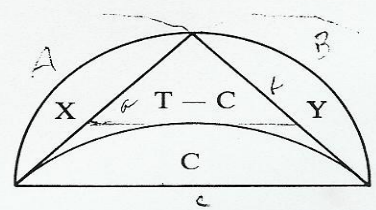
semicircle on a + semicircle on b = semicircle on c

$$(A + X) + (B + Y) = T + X + Y$$
$$A + B + (X + Y) = T + (X + Y)$$
$$A + B = T$$

The last step is a matter of "squaring" a crescent, that is, constructing a square of equal area.

We start again with an isosceles right triangle inscribed in a semicircle. A circular segment, C, similar to segments X and Y, is constructed on the hypotenuse.





The entire semicircle is composed of two small circular segments, X and Y; a similar large segment, C; and $T - C$, the part of triangle "T" which lies above the large segment.

$$\text{semicircle on } c = X + Y + C + (T - C)$$

Each circular segment (X, Y, or C) is proportional to the square of its base (a, b, or c). And since $a^2 + b^2 = c^2$, then

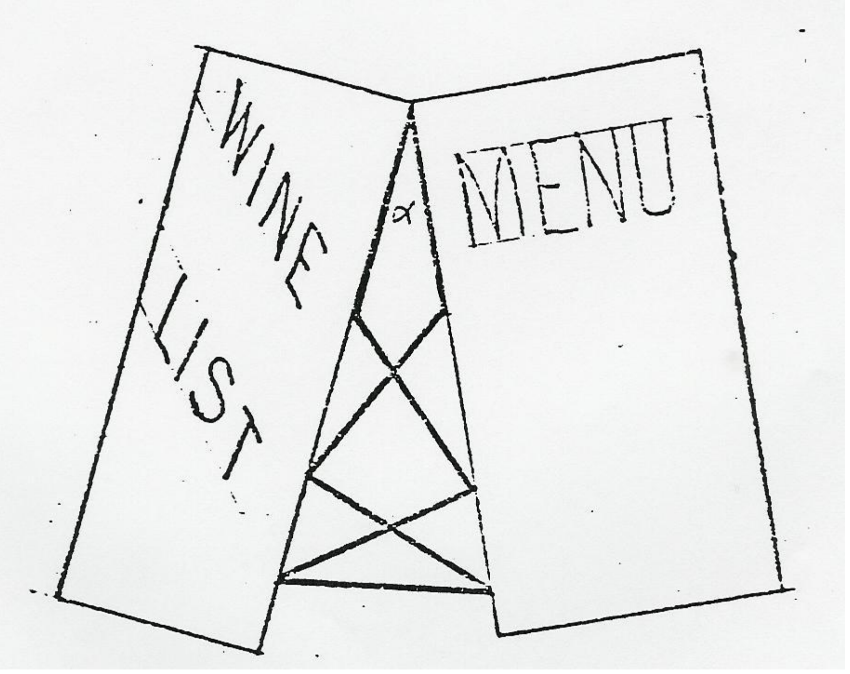
$$\text{segment } X + \text{segment } Y = \text{segment } C$$

If we add $T - C$ to both sides of this last equation, we find that the crescent $X + Y + (T - C)$ equals triangle T.

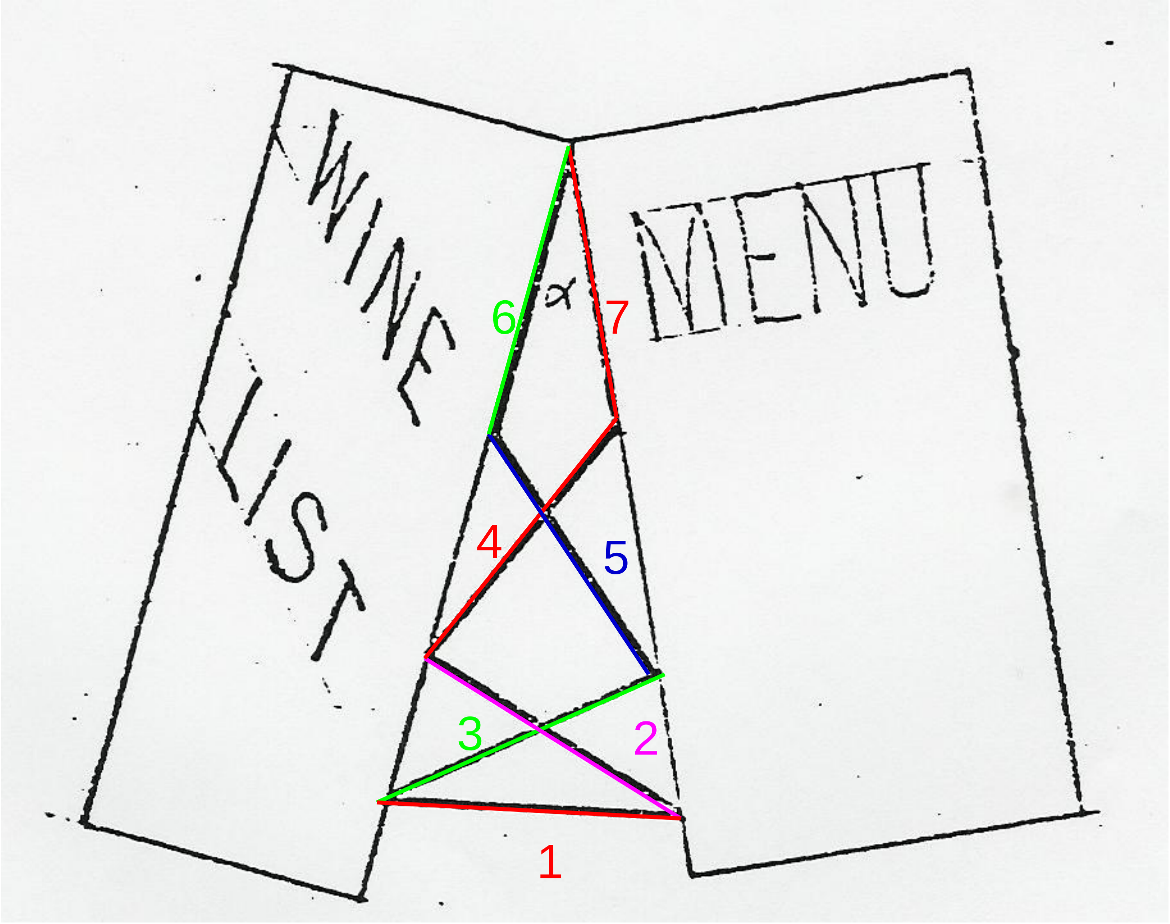
$$X + Y = C$$
$$X + Y + (T - C) = C + (T - C) = T$$

Construction of a Regular Heptagon

Crockett Johnson discovered his construction for the regular heptagon when he observed that if he used his wine list and menus and seven toothpicks arranged as below, the angle formed was equal to π/7, the angle needed to construct a regular heptagon.



Construction of a Regular Heptagon



Heptagon from its Seven Sides

Construction of a Regular Heptagon

Call the ends of a ruler A and Z, and (towards A) place on it a mark X. Draw a line BC the length of AX, and a square BCDE. Erect a perpendicular bisector BC.

With centre C and radius CE draw a circle. Now place the ruler (Fig.1) so that AZ passes through B, with A on the perpendicular bisector of BC, and with X on the circle. Then angle BAC = π/7.

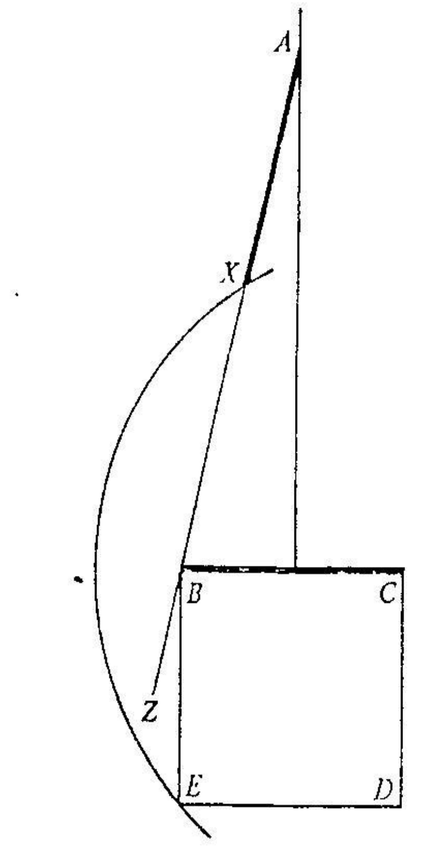
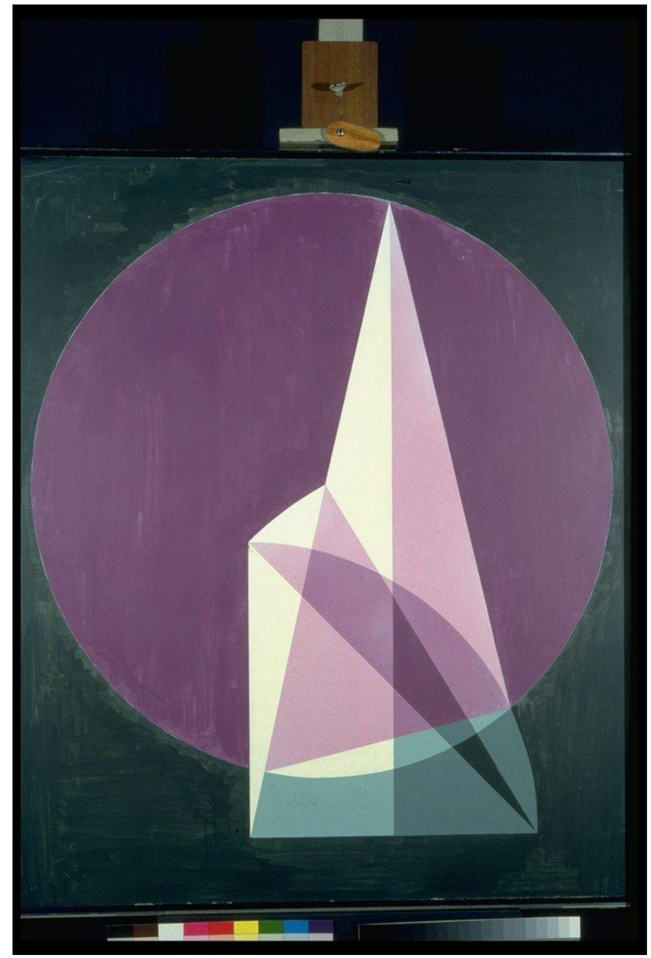
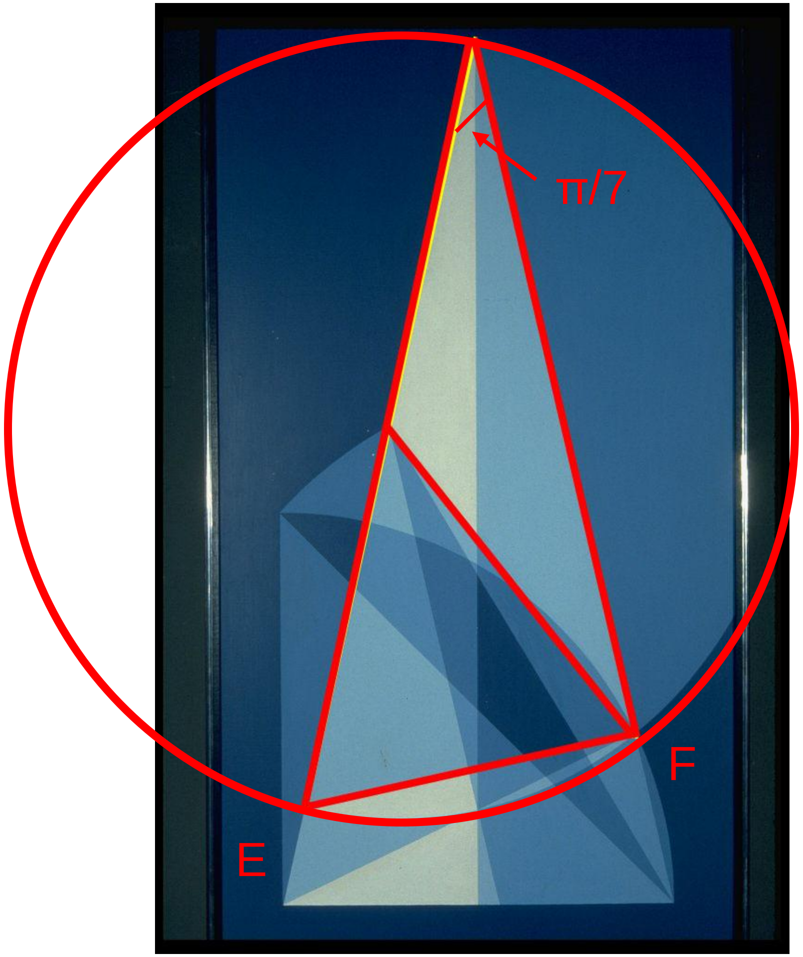


FIGURE 1.

Construction of Hepatagon



Hepatagon



EF is the side of the heptagon.

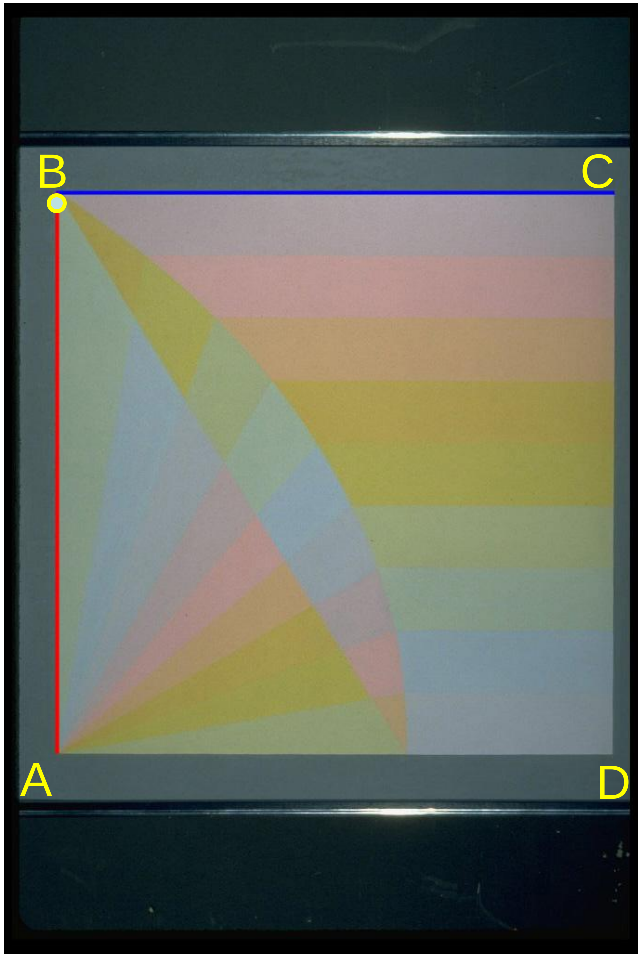
Hippias' Curve

A quadratrix is described by a double motion as follows:

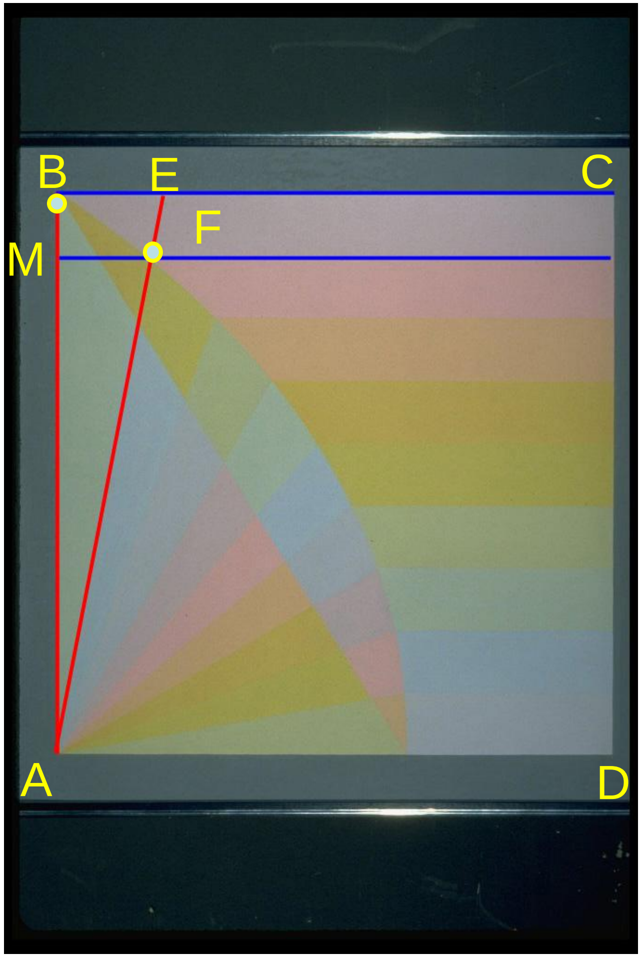
Let a straight line segment AE rotate clockwise about A with a constant velocity from the position AB to the position AD, so that the quadrant BED of a circle is described. At the moment that the radius AE leaves its initial position AB, a line MN leaves BC and moves down with constant velocity towards AD, always remaining parallel to AD. Both these motions are so timed that AE and MN will reach their ultimate position AD at the same moment. Now at any given instant in the simultaneous movement, the rotating radius and the moving straight line will intersect at a point (F is a typical point) The locus of these points of intersection is the quadratrix.

Hippias' Curve

Hippias' Curve

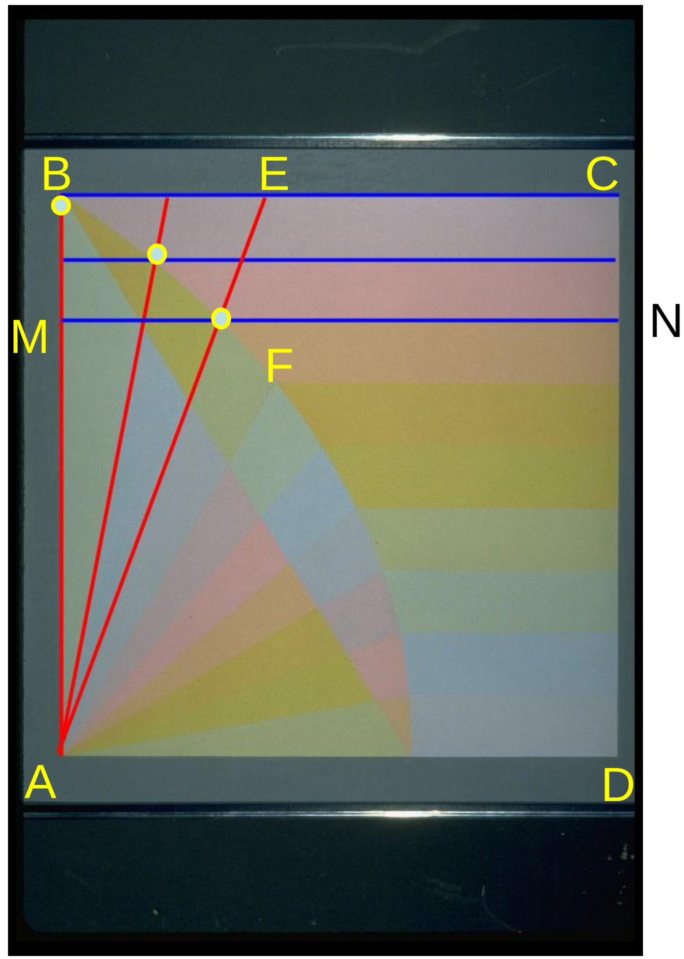


Hippias' Curve

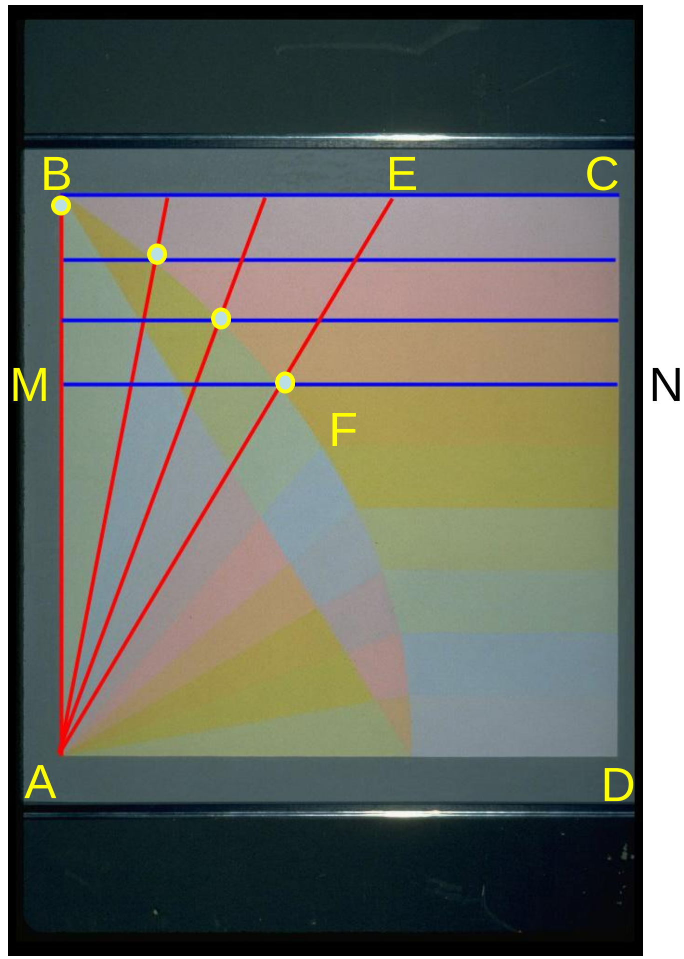


N

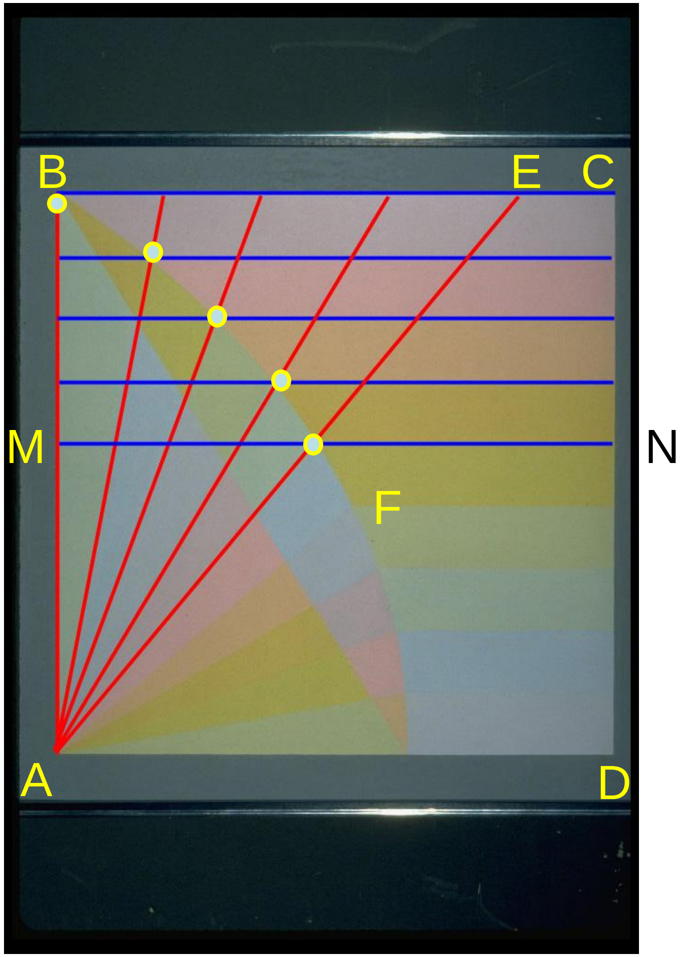
Hippias' Curve



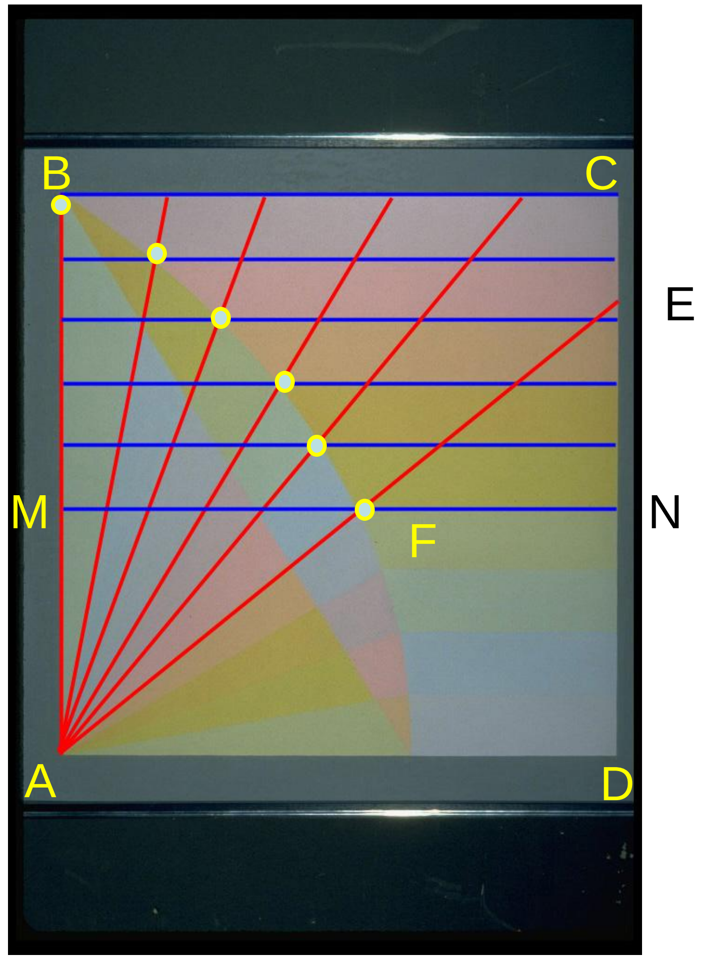
Hippias' Curve



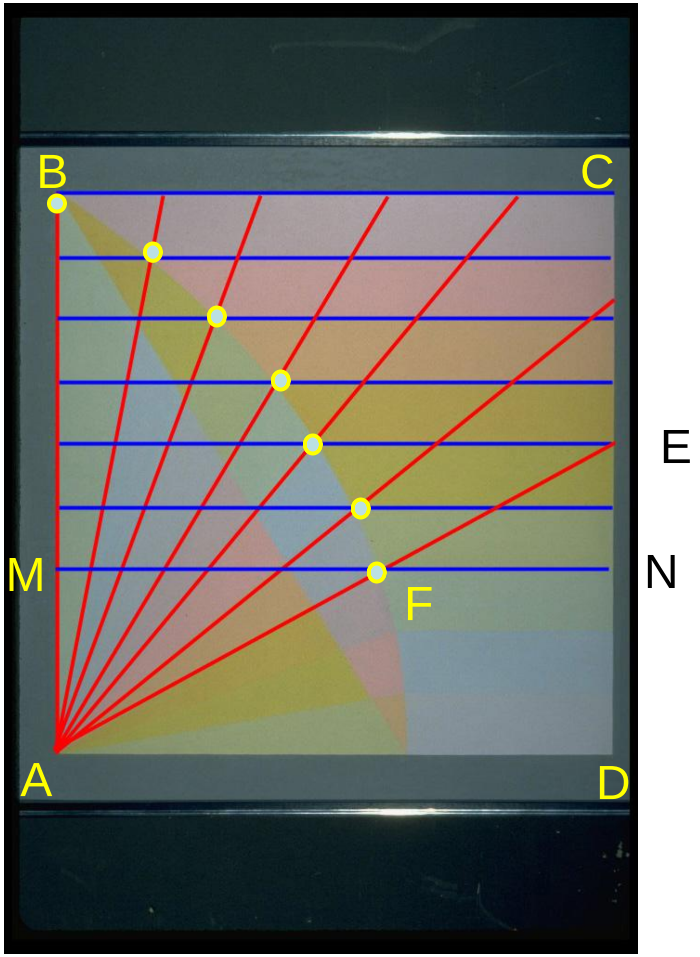
Hippias' Curve



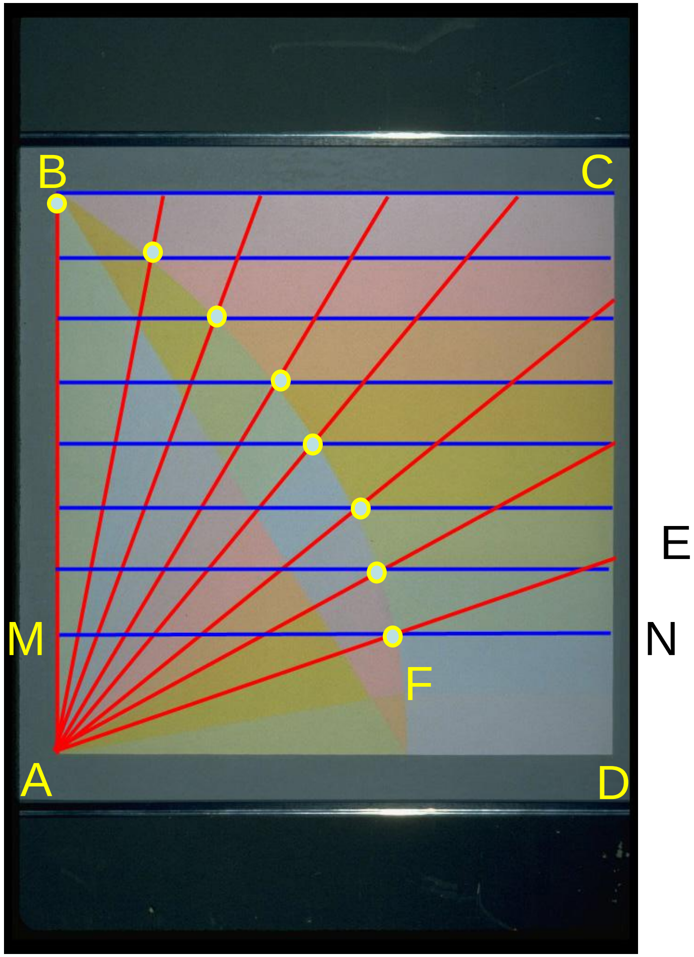
Hippias' Curve



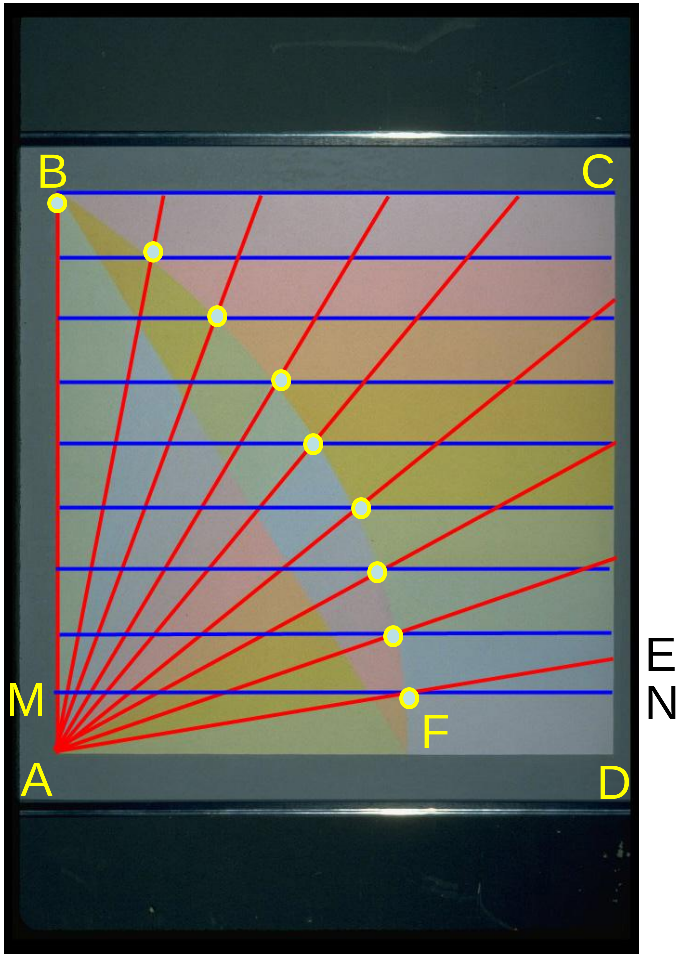
Hippias' Curve



Hippias' Curve



Hippias' Curve



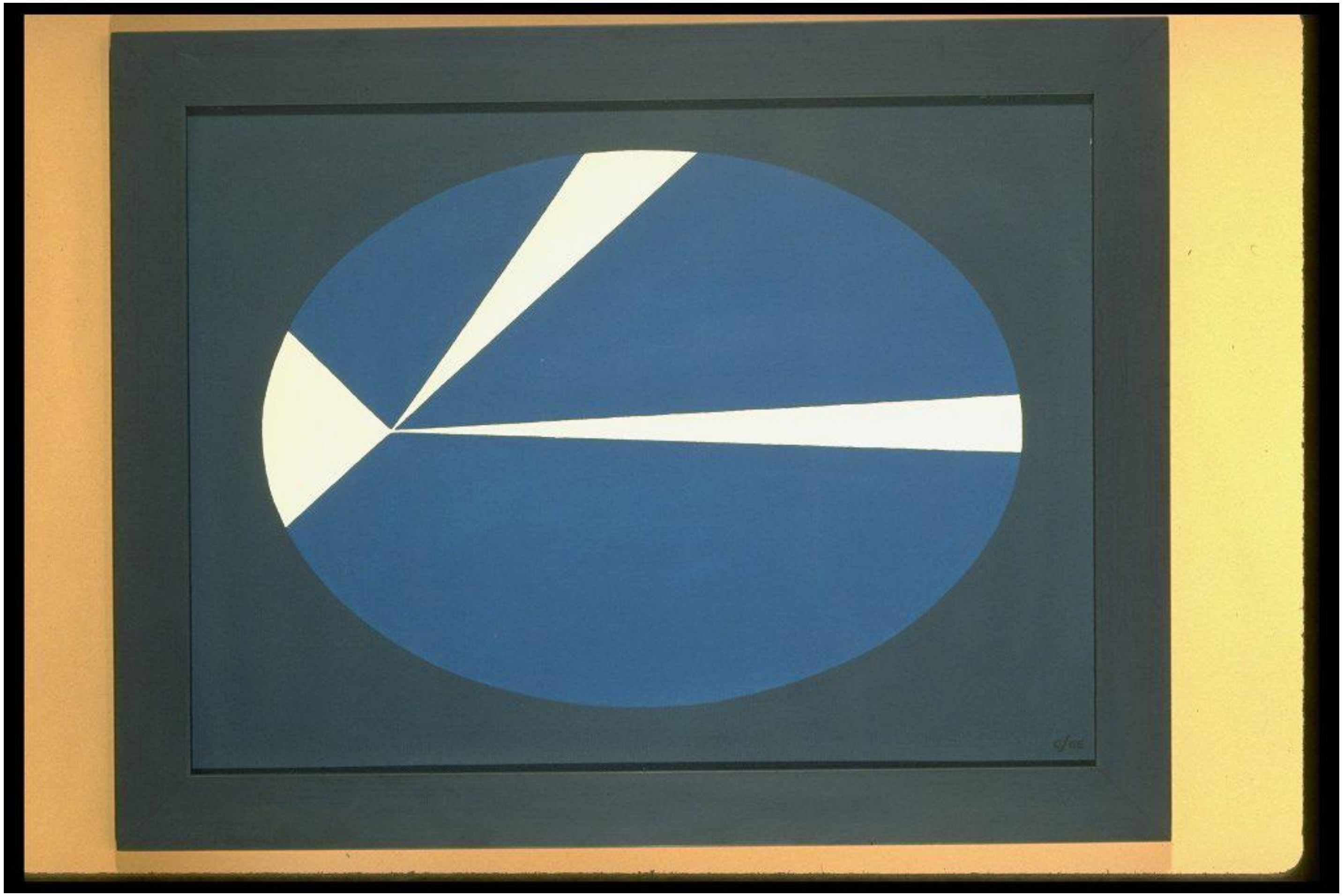
Hippias' Curve

The yellow line is the quadratrix.

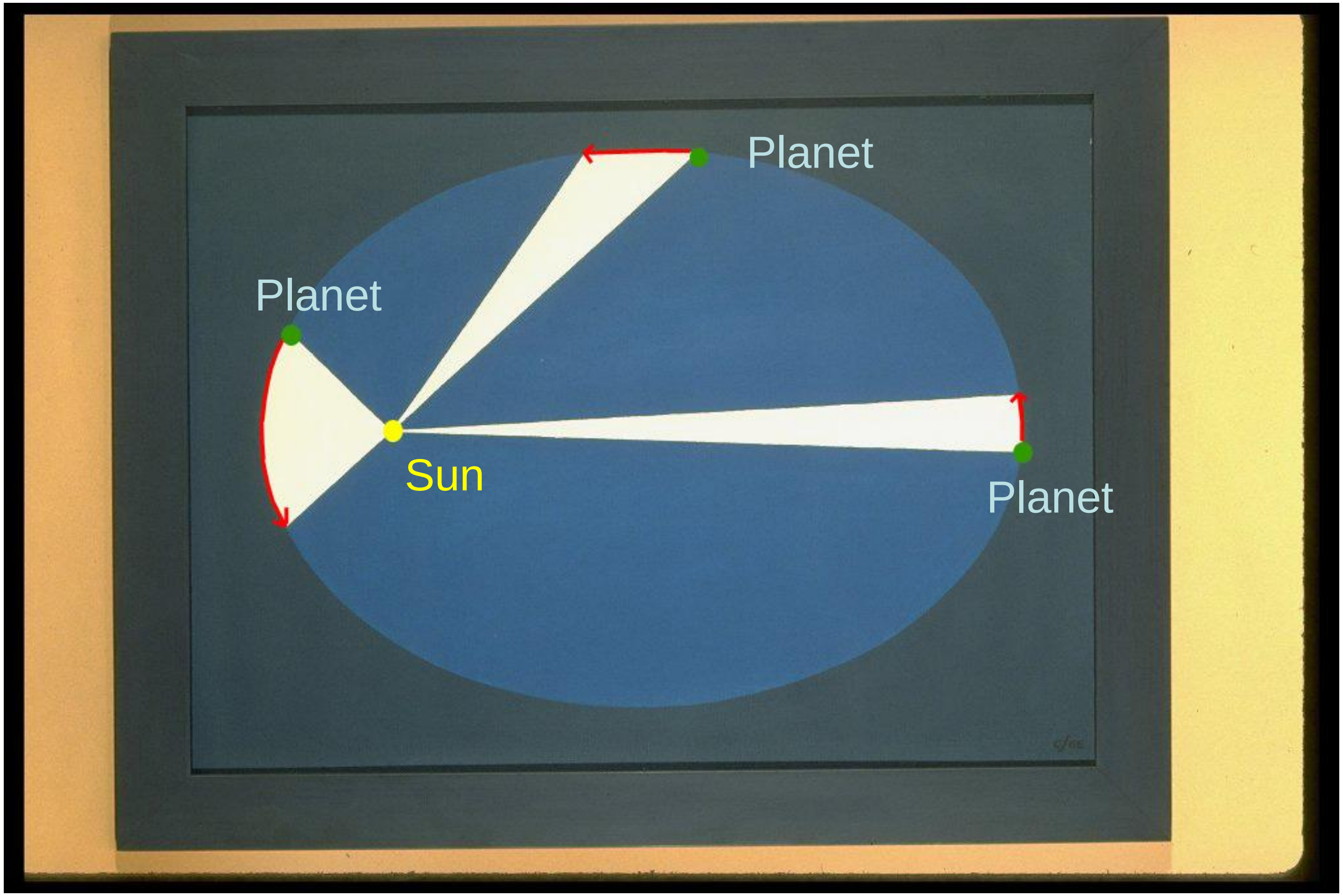
Laws of Orbiting Velocities
(Kepler)

Each planet moves around its orbit, not uniformly, but in such a way that a straight line drawn from the sun to the planet sweeps out equal areas in equal time intervals. The planets move in elliptical orbits with the sun at one focus.

Laws of Orbiting Velocities



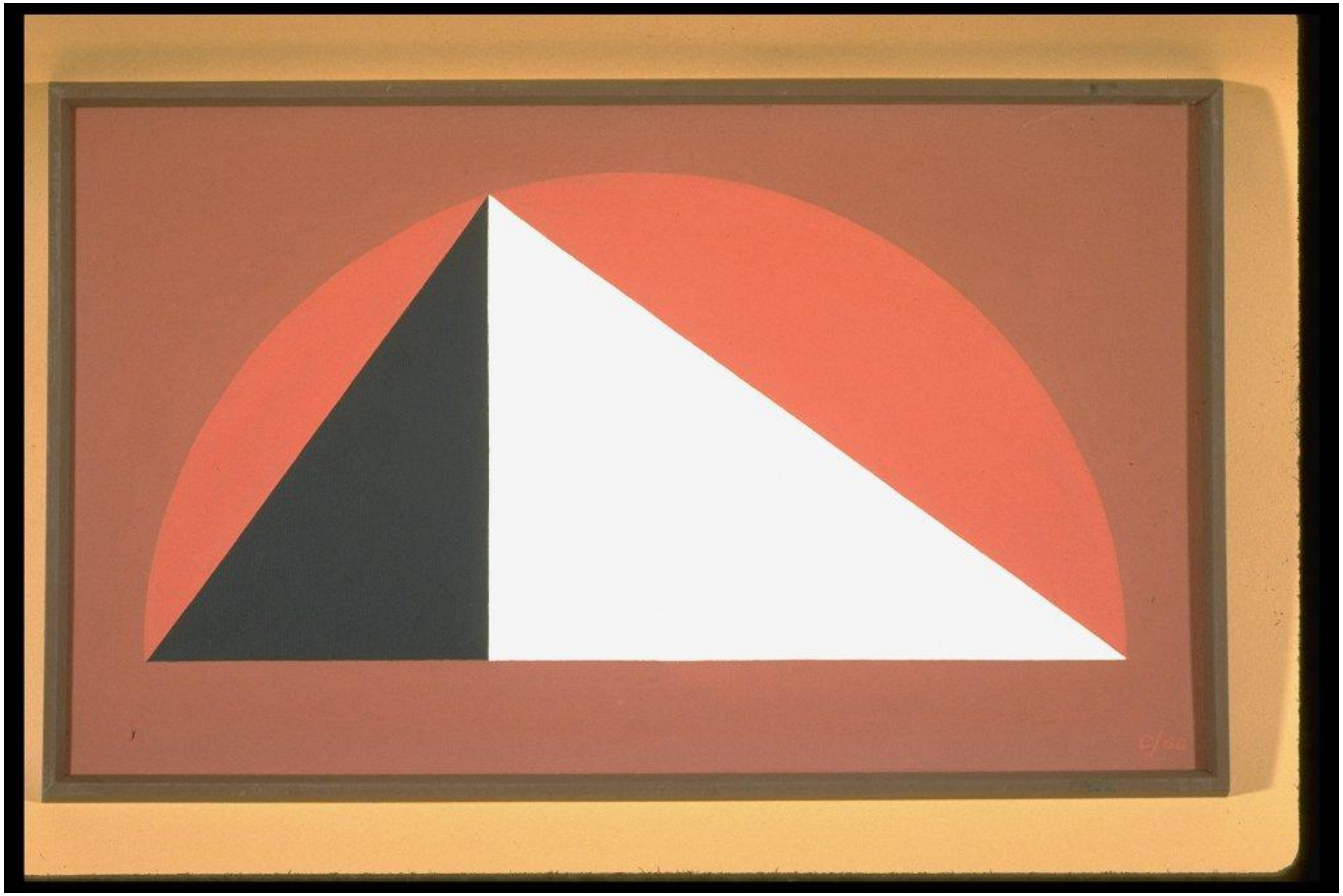
Laws of Orbiting Velocities



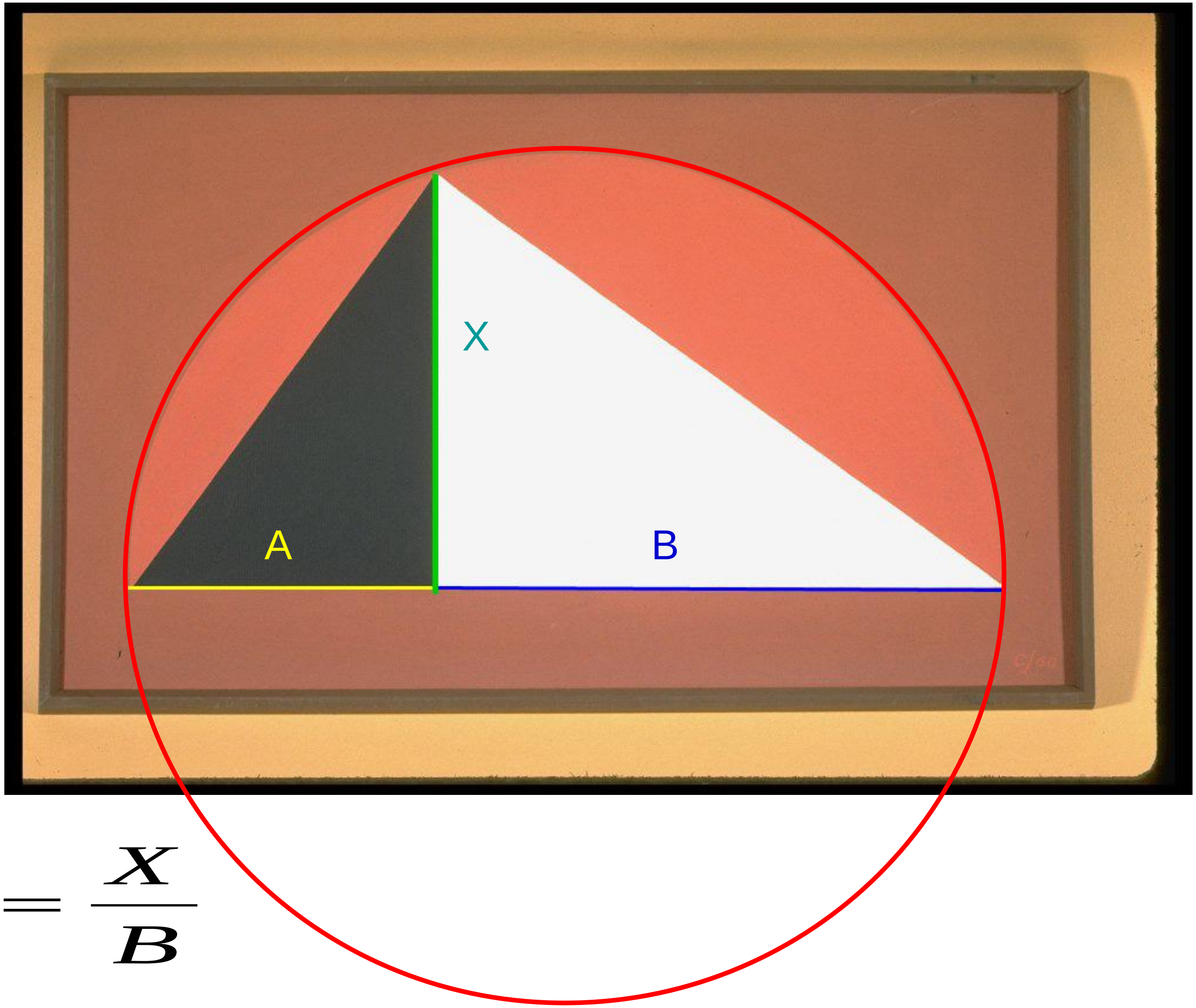
Geometric Mean (Archytas)

- A geometric mean between two lines A and B is a line X such that A / X = X / B.
- Place lines A and B end-to-end and draw a semicircle, using the total lines as the diameter. Erect a perpendicular to the circle from the point where the two original lines join. This perpendicular is the geometric mean between the two lines.

Geometric Mean
(Archytas)



Geometric Mean (Archytas)

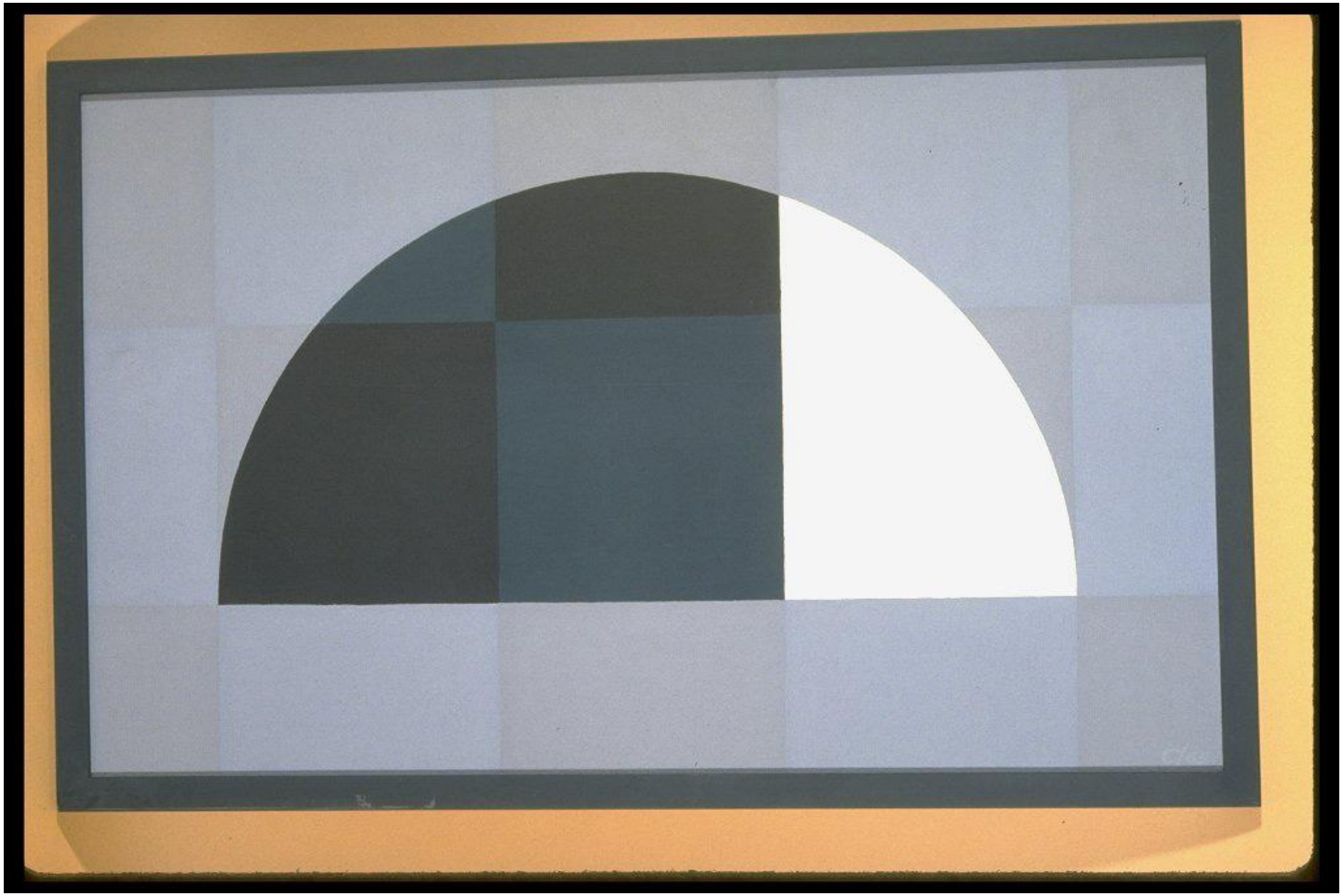


$$\frac{A}{X} = \frac{X}{B}$$

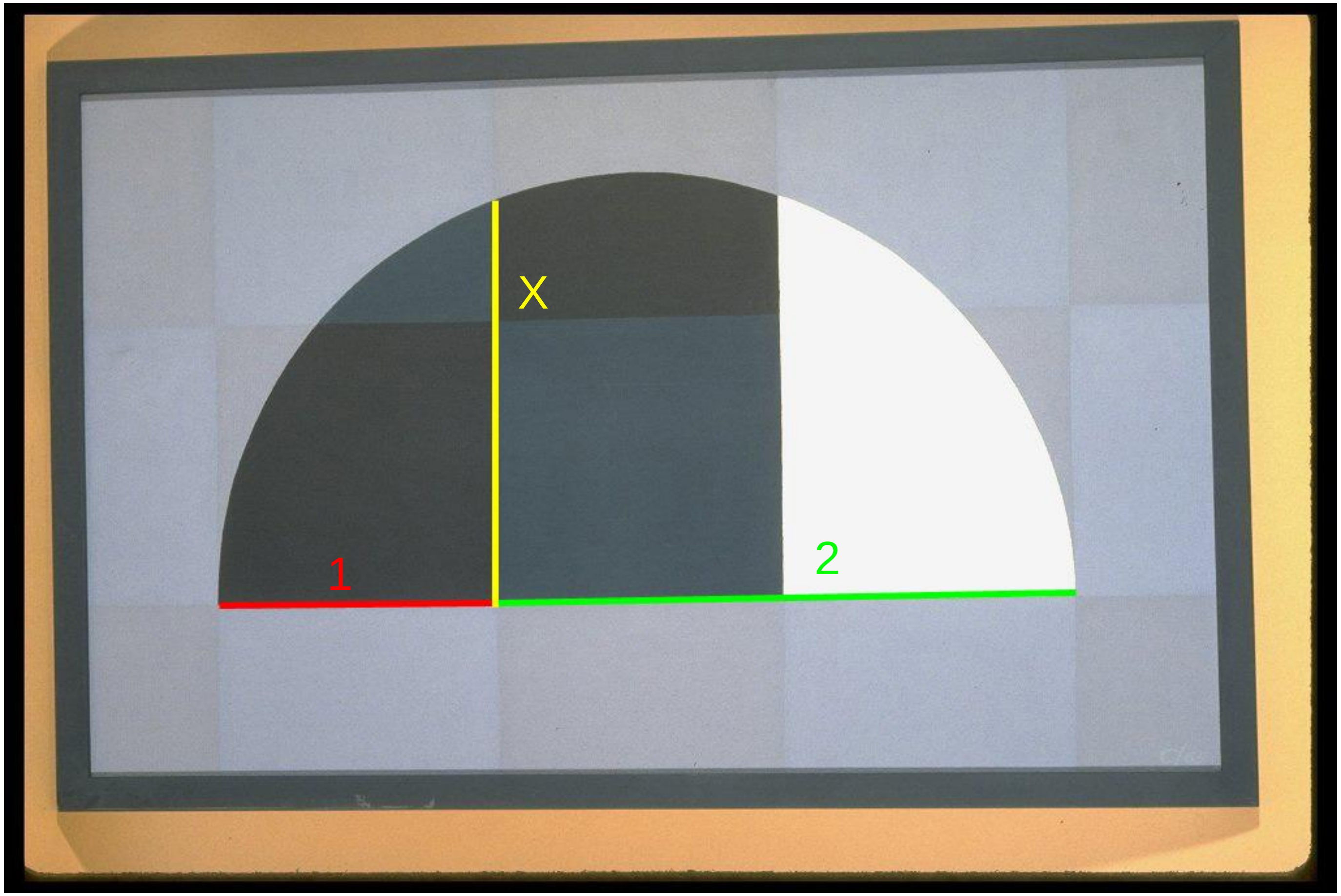
Square Root of Two

- Descartes *La Geometrie*
- “If the square root of GH is desired, I add, along the same straight line, FG equal to unity; then, bisecting FH at K, I describe circle FIH about K as a center, and draw from G a perpendicular and extend it to I, and GI is the required root.”

Square Root of Two
(Descartes)



Square Root of Two



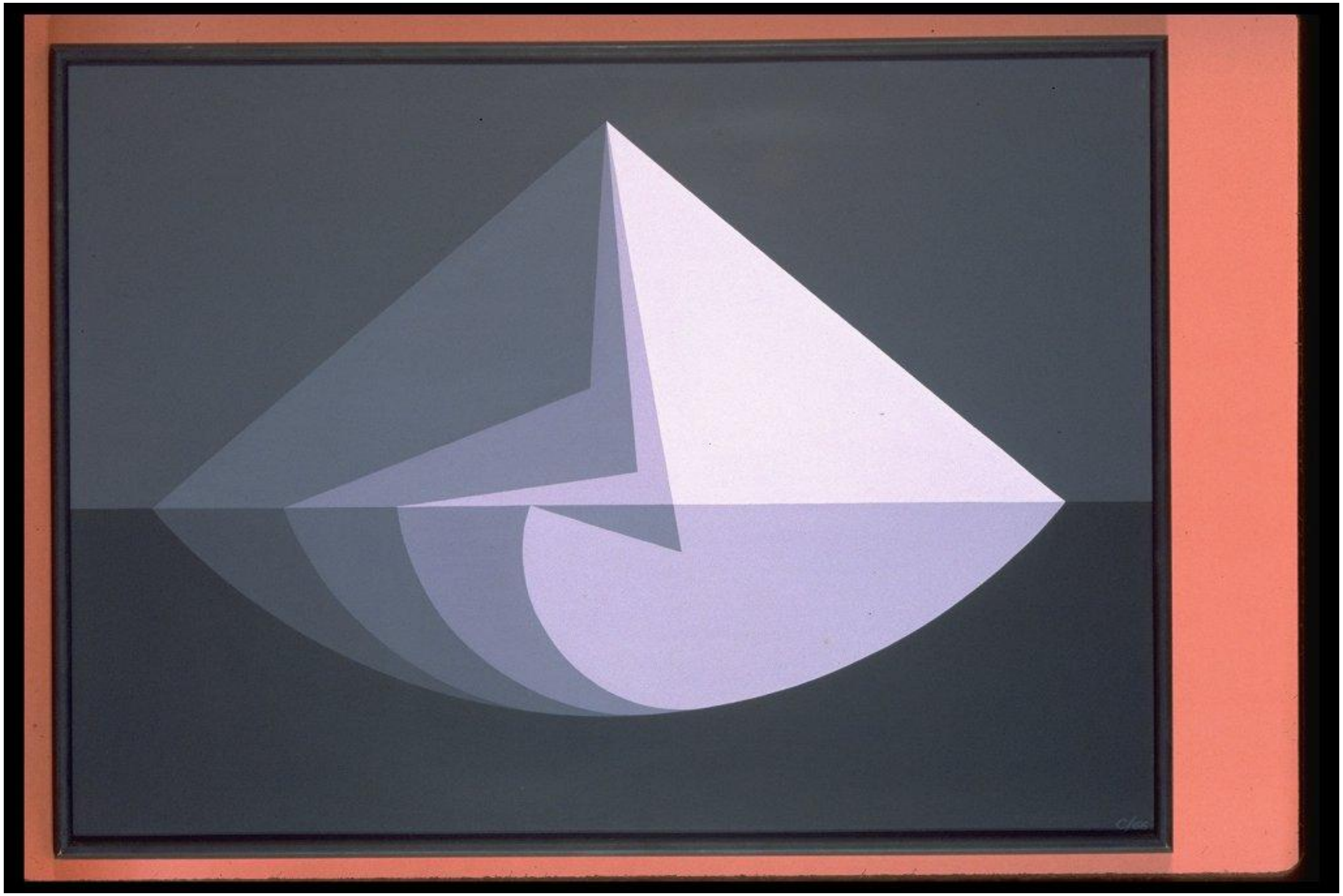
$$\frac{1}{X} = \frac{X}{2} \Rightarrow X^2 = 2 \Rightarrow X = \sqrt{2}$$

Pendulum Motion

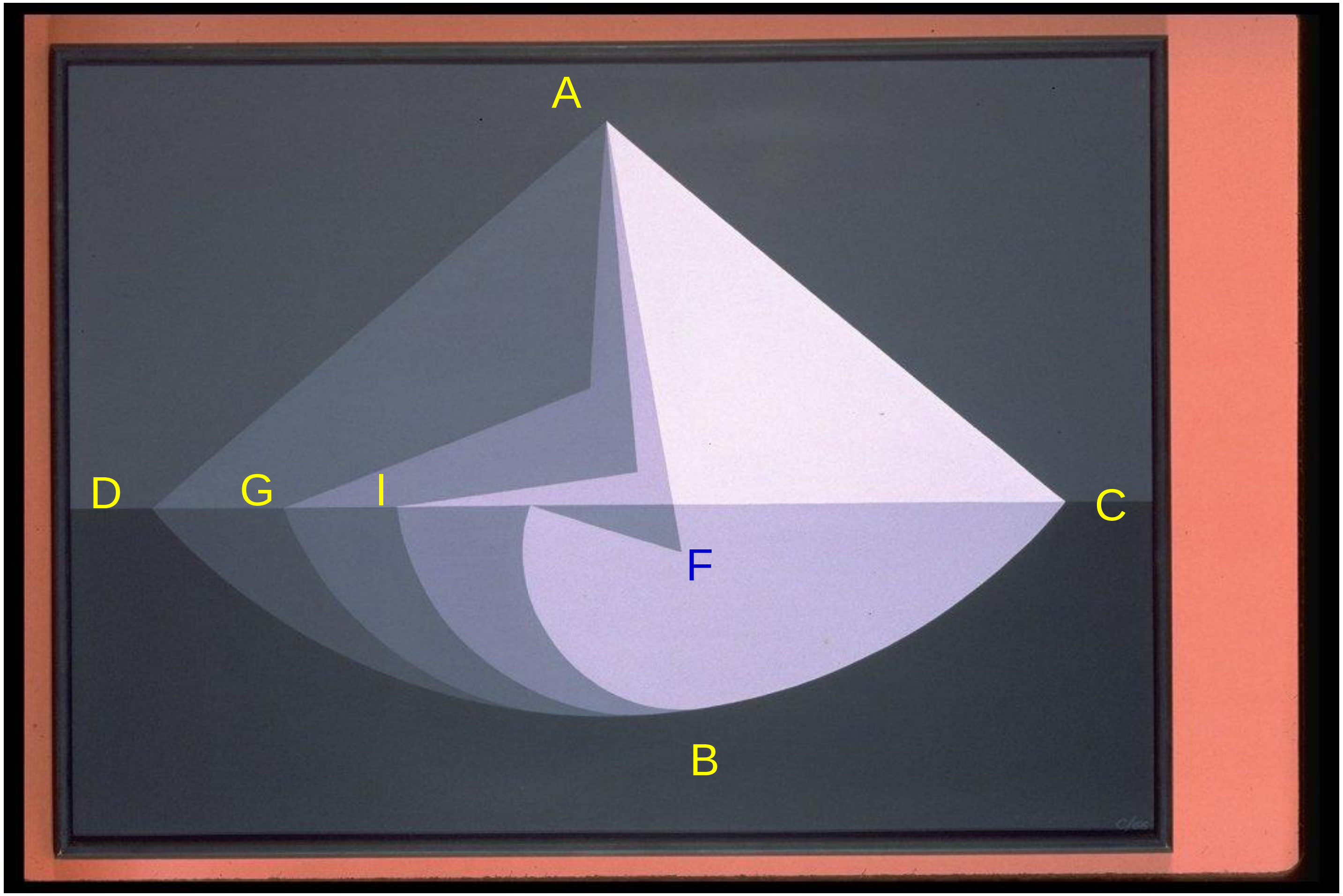
Galileo describe the motion of an object on a string observing that if it starts at C it will tend to rise to D if air resistance is neglected. Should it strike a nail it will still tend to rise to the same height although its path will be different.

Pg 742   The World of Mathematics.

Pendulum Motion



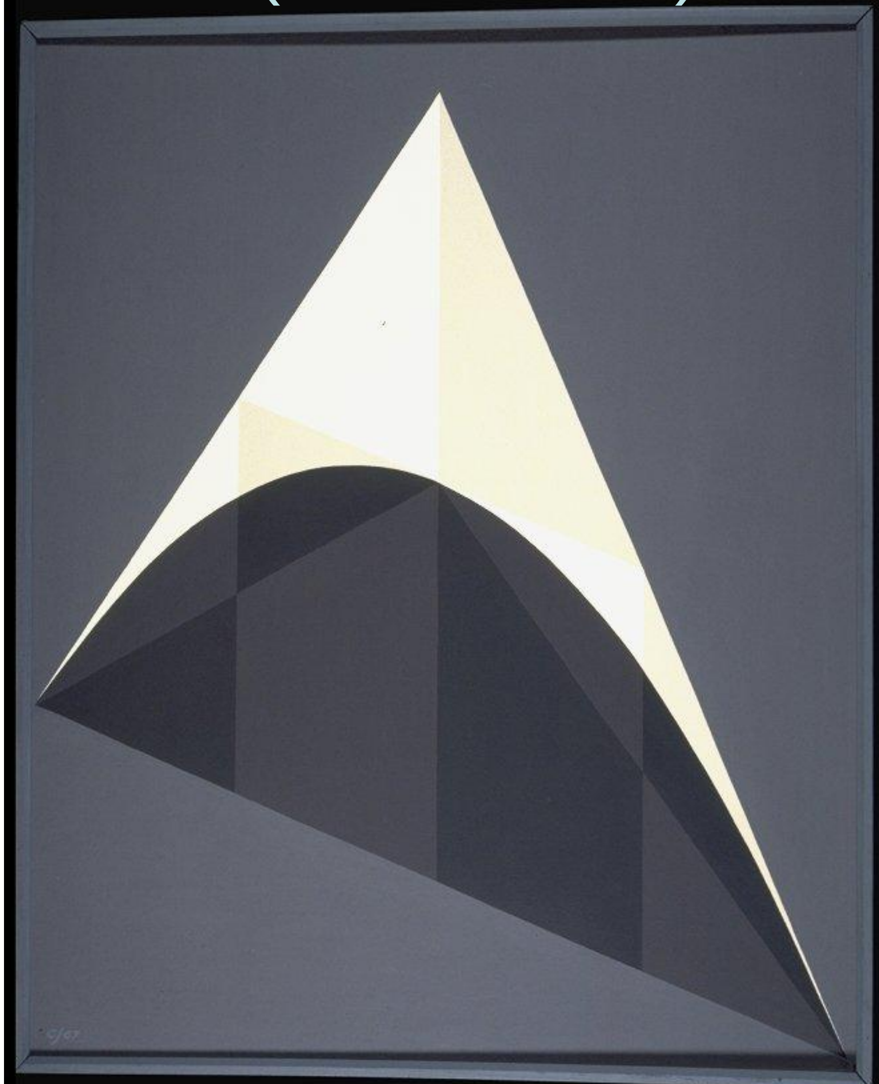
Pendulum Motion



Parabolic Triangles (Archimedes)

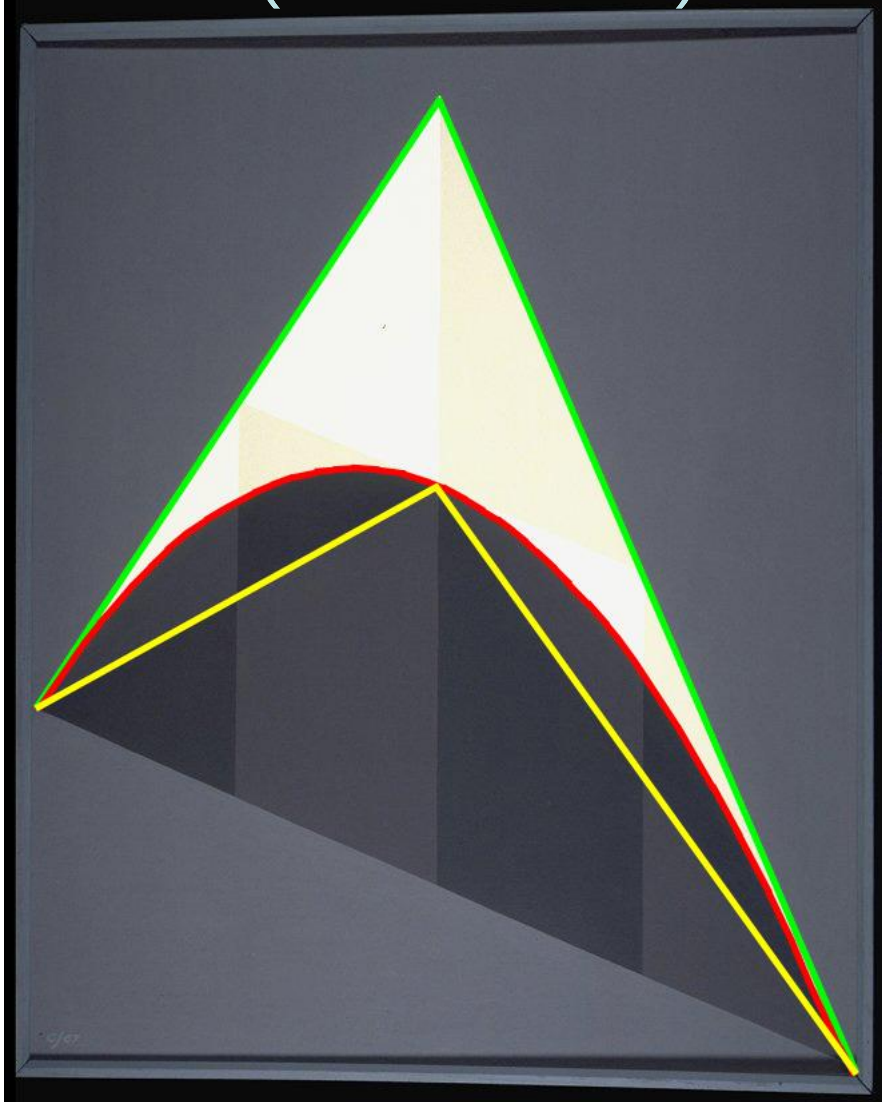
The theorems that Archimedes proved were that the area of the parabolic section was equal to 2/3 the area of the parabolic triangle and 4/3 the area of the inscribed triangle. The parabolic triangle is obtained by drawing the tangents to the parabola at the endpoints of the base of the parabolic section.

Parabolic Triangles (Archimedes)



Parabolic Triangles (Archimedes)

Parabolic area is 2/3 of the area of parabolic triangle 4/3 of the area of the incribed triangle



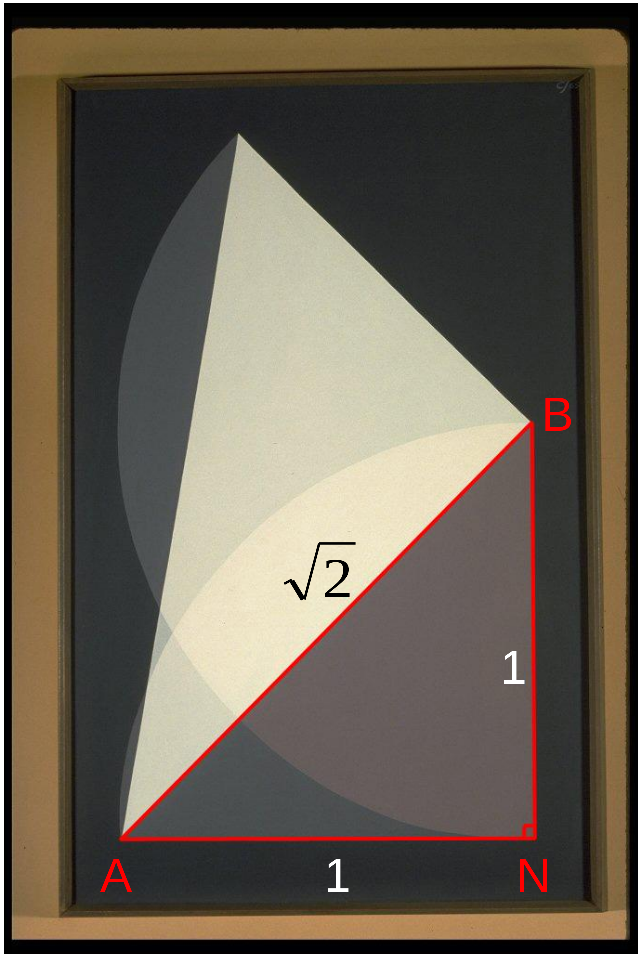
Square Roots of One, Two, and Three

GEOMETRIC NUMBERS Let AN and BN be 1 Then the diagonal AB is the square root of 2 (Pythagorean Theorem) In the large right triangle ABC, if BC = 1 then AC = square root of 3. The compass traces pronounce a statement and also declare its proof.. The square root of 2 is 1.4142.… and the square root of 3 is 1.7321.…. Their decimals run on and on but as produced by the compass and blind straightedge both numbers are quite as finite as 1. The triangle embodies three dimensions of the cube. CB is any edge. AB is a face diagonal, and AC is an internal diagonal.

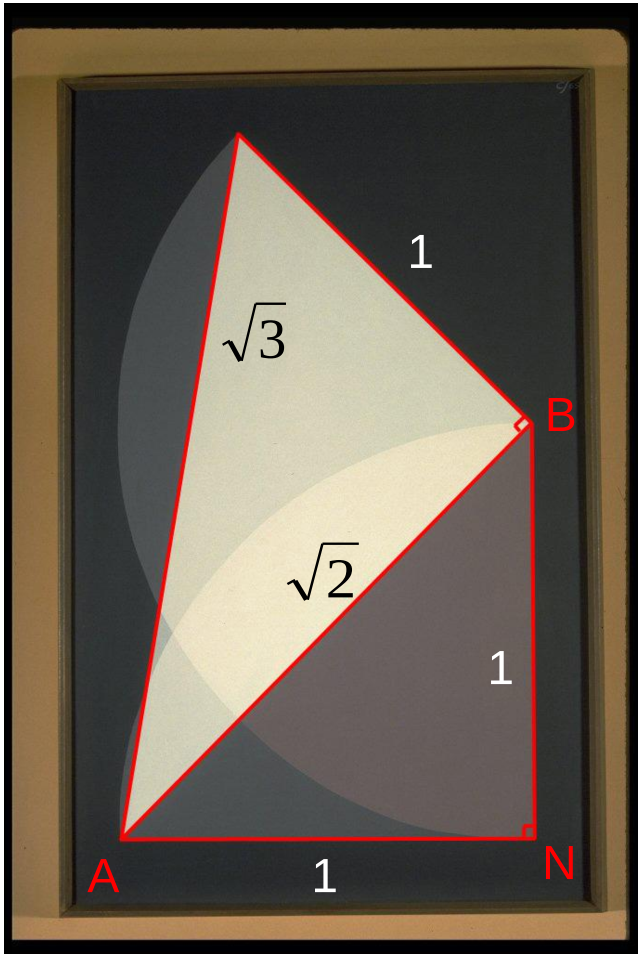
(GEOMETRIC PAINTINGS BY CROCKETT JOHNSON)

Square Roots of One, Two and Three

Square Roots of One, Two and Three



Square Roots of One, Two and Three



Measurement of the Earth (Eratosthenes)

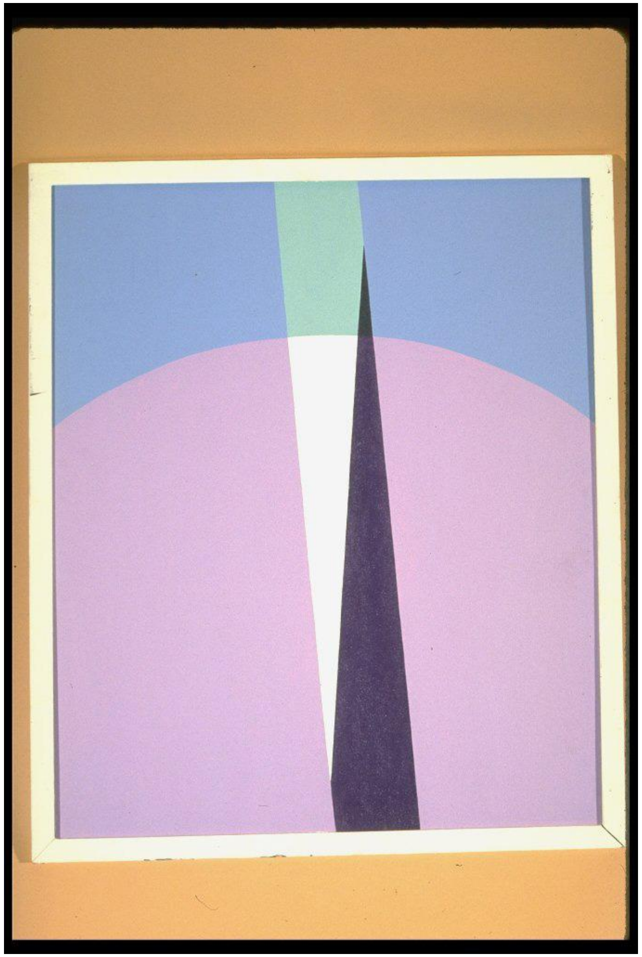
He observed that, while the sun cast no shadow from an upright gnomen in Syene at noon on the summer solstice, the shadow cast at the same at Alexandria… indicated an inclination of the sun's ray with the vertical to be 1/50 of the full circle, that is 7 degrees and 12 minutes. Hence

$$\frac{Circumference}{360} = \frac{5000\, stades}{\frac{360}{50}}$$

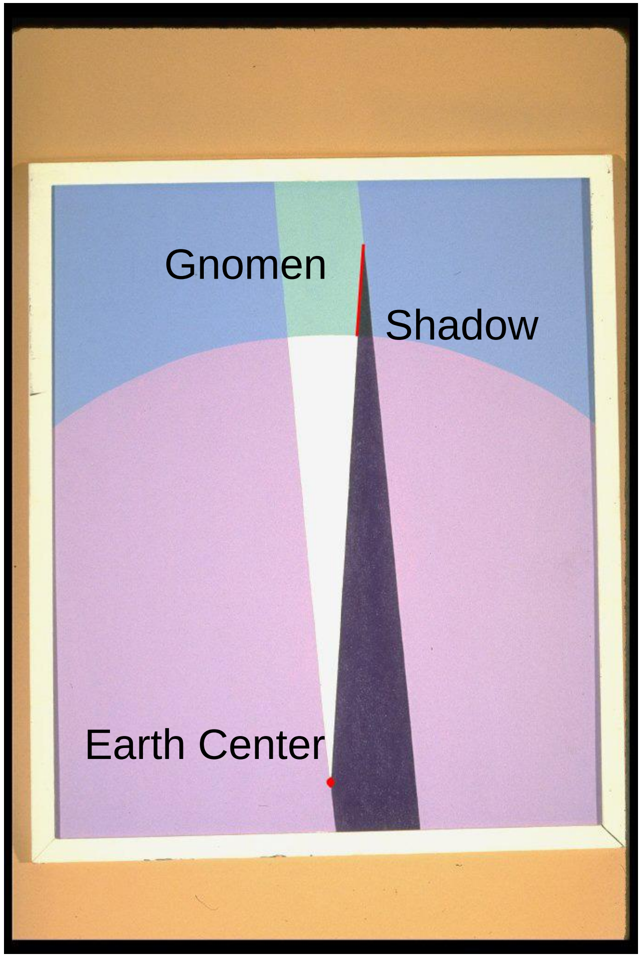
And therefore C=250,000 stades about 25,000 miles.

Calinger, p. 173-174.

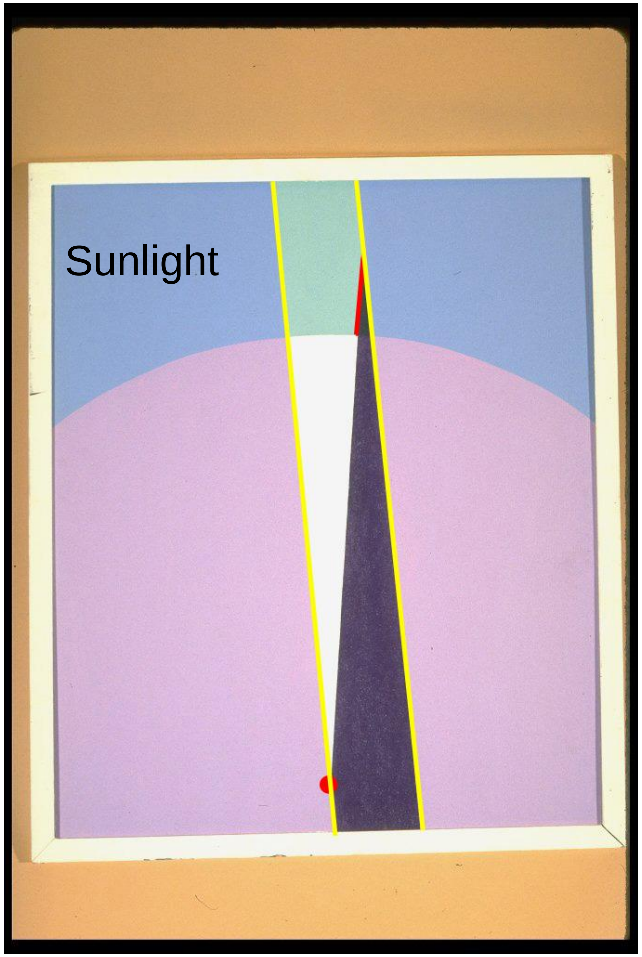
Measurement of the Earth



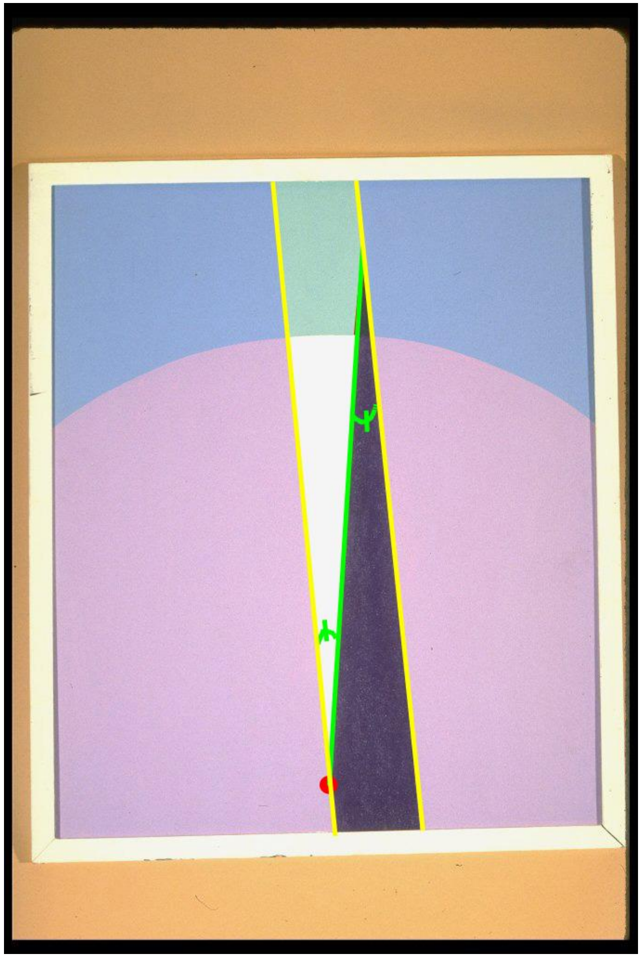
Measurement of the Earth



Measurement of the Earth



Measurement of the Earth



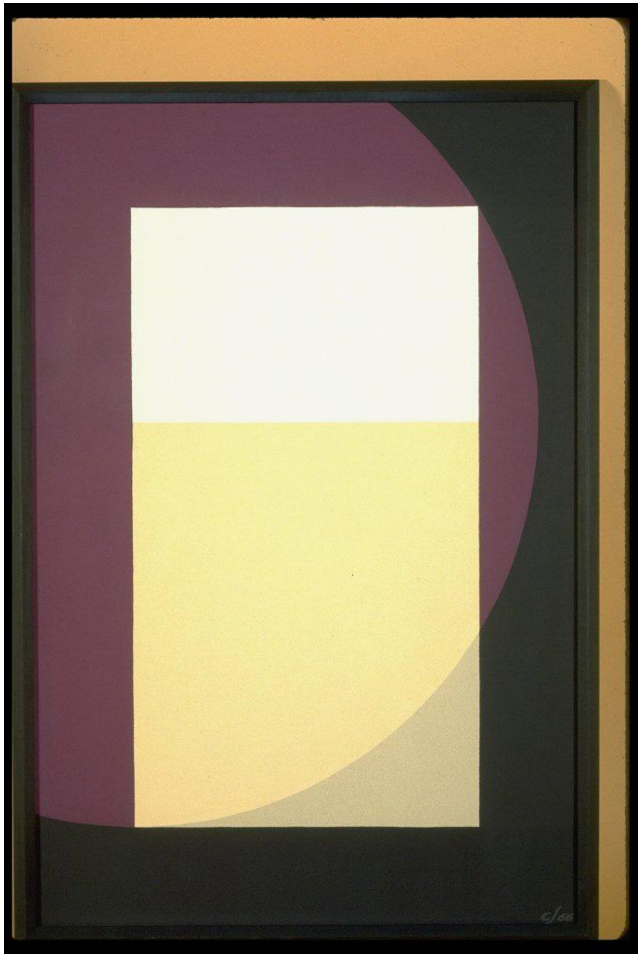
Parallel lines cut by a transversal.

Simple Equation (Descartes)

Finally, if we have $z^2 = az - b^2$, I make NL equal to ½ a and LM equal to b as before: then, instead of joining the points M and N, I draw MQR parallel to LN, and with N as a center describe a circle through L, cutting MQR in the points Q and R; then z, the line sought, is either MQ or MR…..

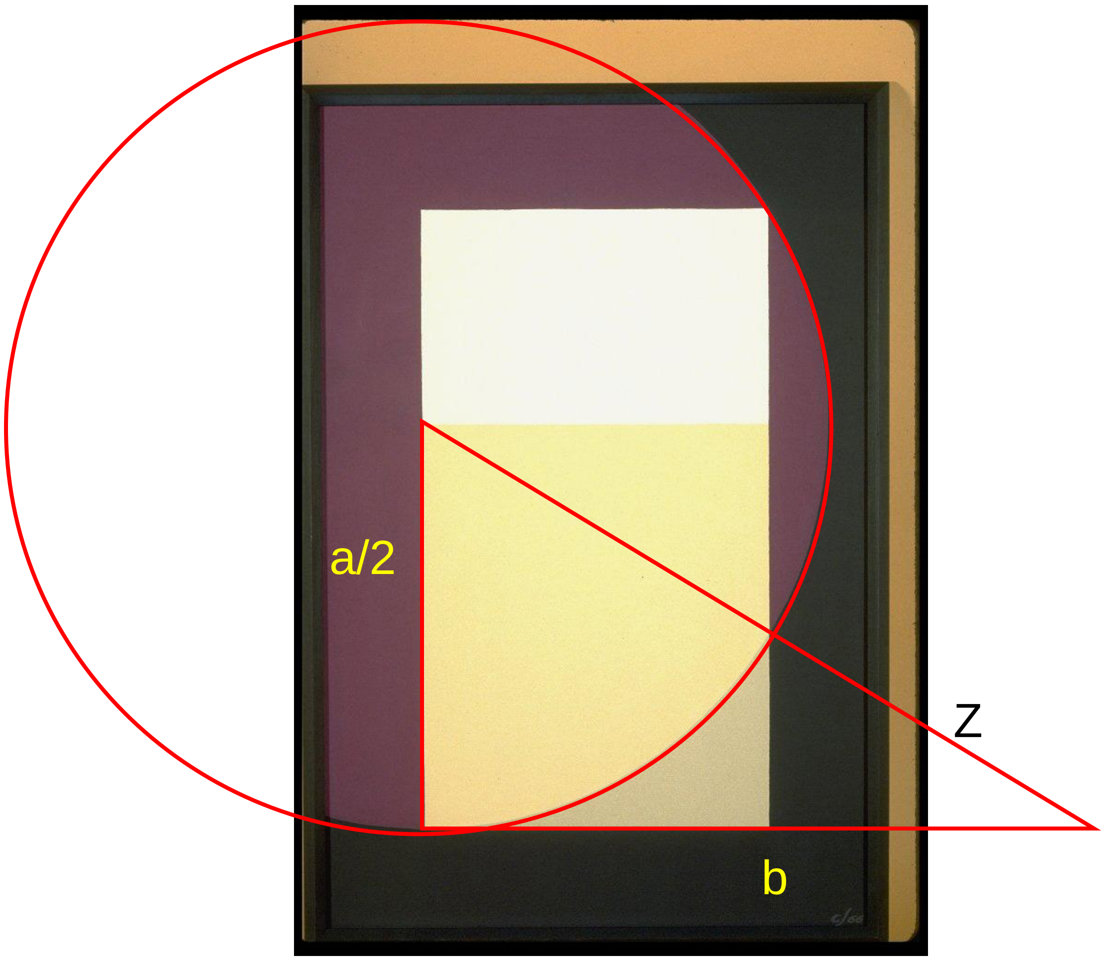
“La Geometrie” by Rene Descartes, *The World of Mathematics*
James Newman, Ed., pg 250-251

Simple Equation
(Descartes)



Simple Equation

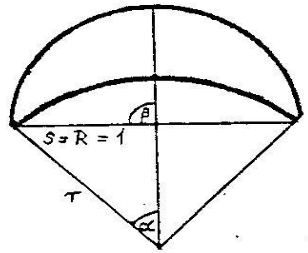
(Descartes)



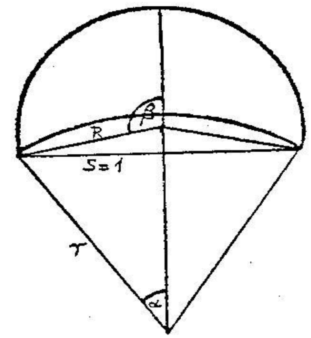
- “A Construction for a regular Heptagon”, *Mathematical Gazette 59* (March 1975): 17-21.

The Five Squarable Circular Lunes

(m:n = 2:1, = 3:1, = 3:2, = 5:1, = 5:3)



2:1

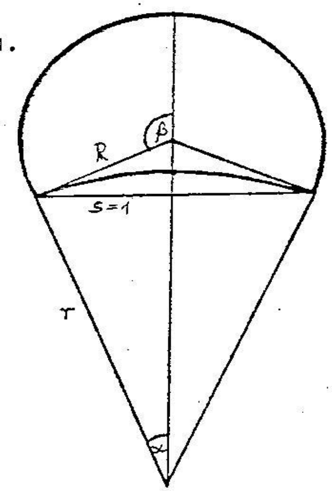


3:1

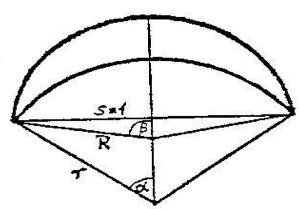
Common chord 2s (=2),
$r \cdot \sin\alpha = R \cdot \sin\beta = s = 1.$
Circulars sectors of
equal area:

$$r^2\alpha = R^2\beta \; ; \quad \Rightarrow$$

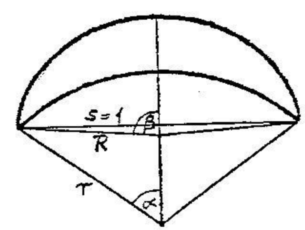
$$\frac{R^2}{r^2} = \frac{\alpha}{\beta} = \frac{\sin^2\alpha}{\sin^2\beta} = \frac{m}{n} \, .$$



5:1



3:2



5:3